PRACTICAL MATHEMATICS

Presenting the Principles of Arithmetic, Equations, Formulas, Mensuration, Graphs, and Logarithms, by a Simple Step-by-Step Method, for Vocational and Home-Study Students, Shop Men, etc.

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THIS IS YOUR FIRST LESSON

READ IT CAREFULLY

To Those Who Have Had Mathematics

No doubt many students have already covered this work in resident school, and the first lessons in mathematics will seem very simple to them. It is well said that everybody has been "exposed" to the fundamentals of mathematics, but it is surprising how few people know the subject so thoroughly that they do the work quickly and accurately. Therefore, it will pay the student to review the things that he may know; and if he knows them fairly well, it will take a surprisingly short time to make this review.

How to Study Mathematics

Every good teacher has a definite plan for carrying on his work, and he leads the student step by step through this plan, although the student may not be conscious of the plan. However, since you and your instructor will not be meeting each other personally, we want to tell you something about our plan for teaching you mathematics.

For the average student it does not pay to make one of the sections of the book a lesson. This is usually too much material, and it is difficult to learn all of this at one time. Therefore, we have divided each section into lessons. If you will study each lesson separately and make sure that you understand it before you start on the next lesson, you will have the satisfaction of knowing that small unit, and then you can build upon it the next unit, and so on.

Each lesson has a definite aim which is presented in four steps. Each step is necessary; if you slight or omit one of them you will fail to get all the lesson has to offer.

Step 1—The preparation or background
Step 2—The presentation of the new material
Step 3—The application or trial—working the Illustrative Examples
Step 4—The test—the Practice Problems

Step 1—Preparation

The first step deals with the very necessary problem of creating a background for the new material that you are going to study. It is a simple

truth that no one can learn any new material unless he can attach it to something he already knows; this is the reason for the preparation step.

In the preparation step in each lesson we want you to recall what you already know about the subject. You would do this almost unconsciously, but if you will spend a few minutes thinking about what you know of the subject it will help you in learning the new material. Sometimes it will be necessary to review something which you already have been taught in order to get a good foundation on which to build the new material.

Step 2—Learning the New Material

The second step of each lesson is learning the new material, and this requires very careful study. Do not hurry. Pay particular attention to definitions of new words and terms. Try to discover the important or key processes of each lesson. This step teaches you how to reason out the examples and problems which are to follow. Keep in mind that if you are able to **think through** and **reason out** a problem, the answer will come of itself.

Step 3—The Application or Trial

The third step is the application or trial by working the Illustrative Examples. **Study each Illustrative Example step by step as you read it.** After you have studied the solution step by step, **lay the book aside and work the example on paper.** When you have finished it, open the book and compare your work with the text. This is one sure way of checking what you know. Therefore, work every example, even though it almost amounts to copying the material, so that you will get some action into your thinking. On completion of this step it is a good plan to review the governing principles which were set forth in the beginning of the lesson.

Step 4—The Test

The fourth step is a test on what you have learned, and for this step you work the Practice Problems. You should be able to work these problems accurately and with fair speed.

If you have a feeling of uncertainty as to how a problem should be worked, this means that you have not understood what you read in Step 2. **Go back to the rule or principle which covers the problem** and do some more studying before you try to work it. You may feel that this is going to take too much time, but in reality it saves time because the more thorough you are, the better your working knowledge of the subject which you are studying will be, and this will save time later.

This is perhaps the most important step in your study. It is the working of the problems that tests whether you know the subject or not. If you find that you do not know it, review Steps 2 and 3.

The Examinations

At the end of each section of the book you will find two examinations: the Trial Examination and the Final Examination. You are to work the Trial Examination first—without referring to the Instruction Book. Then turn to the Solutions to Trial Examination Problems. With these you will be able to check your own solutions step by step so that you can determine whether any errors are because of inaccuracies in calculations or because you have failed to grasp the principles involved. If any problems are wrong, you should review the section carefully. If your problems are all correct, you are ready for the Final Examination. After you have completed the Final Examination, turn to your Instruction Book and check your answers with those given in the back of the book.

PRACTICAL MATHEMATICS

Section 1

Lesson 1

For Step 1, read the first section, "Reading and Writing Numbers," and think of what it says. For Step 2, learn the Arabic notation and the Roman notation. For Step 3, work Practice Problems 1 to 4. For Step 4, work Practice Problems 5 to 10.

READING AND WRITING NUMBERS

Numbers play an important part in the everyday affairs of everybody. Ever since the beginning of the human race, people have been asking the questions: How much? How many? How far? How long?—and so on, which showed the need for numbers with which to answer these questions properly.

As a rule, very large numbers are not needed in the daily work, but in papers, magazines, and books, large numbers are often found which one should be able to read and understand easily.

There are two common methods used for calculations and notations. The **Arabic notation** is generally used in calculations, while the **Roman notation** is used for numbering chapters in books, on clock faces, etc.

In the **Arabic notation** ten figures are used as the symbols. These are given below with their names. These ten figures, or digits as they are sometimes called, can be used in any combination, using one or more at a time as may be necessary.

0	1	2	3	4	5	6	7	8	9
cipher	one	two	three	four	five	six	seven	eight	nine

All numbers are expressed by the above ten figures or digits. This includes the cipher. The cipher, which is sometimes called naught or zero, has no value alone but merely fills vacant places.

A figure or digit has different values according to its location or the place it occupies.

The word **place** in numbers has a special meaning as shown in Table I, which gives the names of the different places in numbers.

TABLE I

Trillions Period			Billions Period			Millions Period			Thousands Period			Units Period		
Hundreds Place	Tens Place	Units Place	Hundreds Place	Tens Place	Units Place	Hundreds Place	Tens Place	Units Place	Hundreds Place	Tens Place	Units Place	Hundreds Place	Tens Place	Units Place
7	8	9,	6	4	3,	5	4	9,	2	7	6,	8	5	1

Each group of three figures is called a **period** and is separated from the other periods by a comma. This makes it easier to read large numbers.

The number shown in Table I is read as follows: seven hundred eighty-nine trillion, six hundred forty-three billion, five hundred forty-nine million, two hundred seventy-six thousand, eight hundred fifty-one. Read this over carefully so that you understand it thoroughly.

The period at the right end with the figures 851 is called the units period; the second period with the figures 276 is called the thousands period; the next three consecutive periods are called the millions, billions, trillions. The higher numbers are not shown in Table I. Ordinarily one does not see higher numbers than those used for war debts, which run into billions of dollars.

You will notice that each period of the figures is read the same way, then the name of that period is put at the end. These period names indicate the location of the figures in regard to the units place. For instance, take the number 125,000,000,001. It is read, one hundred twenty-five billion, one. Nothing is said about the ciphers which merely fill vacant spaces but locate the figures 125 in the billion period.

The following are examples of numbers and how to read them. Always read each period as if it were alone, then read the next

to the right alone, and so on to the end. **Do not use the word "and" in reading such numbers.**

10,067,072—Ten million, sixty-seven thousand, seventy-two
1,003,500—One million, three thousand, five hundred
800,010—Eight hundred thousand, ten
87,109—Eighty-seven thousand, one hundred nine
6,000—Six thousand
4,120—Four thousand, one hundred twenty

In the **Roman notation** letters are used as the symbols instead of figures. The letters used and their values are as follows:

The numbers from 1 to 9 are written thus:

I II III IV or IIII V VI VII VIII IX

The tens are written thus:

Letters:	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
Values:	10	20	30	40	50	60	70	80	90

The hundreds are written thus:

Letters:	C	CC	CCC	CD	D	DC	DCC	DCCC	DCCCC or CM
Values:	100	200	300	400	500	600	700	800	900

The thousands are represented by the letter M.

When a letter is followed by the same letter, or by one less in value, add the values of the letters. Thus: XX=20.

When a letter is followed by another greater in value, subtract the smaller from the larger. Thus: IX=9.

When a letter is placed between two letters greater in value, subtract the smaller from the sum of the other two. Thus: XIV=14.

$\overline{\text{X}}$ means ten thousand. $\overline{\text{V}}$ means five thousand. The line placed above the letter increases its value a thousand times.

The following are examples of the above principles:

XI = 11	XVI = 16	XLVI = 46
XII = 12	XVII = 17	LXXIV = 74
XIII = 13	XVIII = 18	XCVII = 97
XIV = 14	XIX = 19	CCLXIX = 269
XV = 15	XXXIII = 33	CCCLII = 352

MDCCCCXLI or MCMXLI = 1941

PRACTICE PROBLEMS

Write in words:

1. 18,765,972.
2. 834,769,780.
3. 3,576,879,421.
4. 10,805,056. Ans. Ten million, eight hundred five thousand, fifty-six.

Write in figures:

5. Seventy-eight million, forty-one thousand, seven.
6. One thousand three. Ans. 1,003.
7. Five hundred six thousand.
8. Ninety million, two thousand, three hundred twenty-seven.
9. Three hundred five thousand, seventy-nine.
10. Eight hundred sixty-four million, four thousand, twenty.

Lesson 2

For Step 1, think of some fact which gives a background to the material in the lesson. Addition is putting things together, and the first three paragraphs of the lesson give a very good example. For Step 2, learn the method of adding numbers as explained in this lesson. For Step 3, work the Illustrative Examples without referring to the book. For Step 4, work the Practice Problems.

ADDITION

Uniting two or more numbers to make one number is called **addition.** The result obtained from adding two or more numbers is called the **sum.** The sign + (read plus) is used to indicate addition.

Addition is, no doubt, the most common operation in mathematics. Everyone knows how to add money. If you have $17.00 in your purse and earn $7.00 more, by the addition of $17.00 and $7.00, you find that the total amount is $24.00.

Only quantities of the same kind can be added. Thus you cannot add dollars and fence posts or dollars and cents together by simply adding the numbers. The addition of 205 dollars and 101 cents would give the sum of 306, but this would be neither dollars nor cents. Neither can you add inches to pounds nor oranges to apples.

Numbers to be added must be placed so that their corresponding figures will be in vertical columns; placing all the **units** from each

of the numbers in the **units** column, all the **tens** in the **tens** column, all the **hundreds** in the **hundreds** column, and so on. This is shown in the Illustrative Examples.

When you add real things, the answer should contain the **name** of the things added. Thus, 10 nails+8 nails=18 **nails;** 7 yards+5 yards=12 **yards.** If this idea is carried out in addition, subtraction, multiplication, and division, many errors may be avoided in mathematics.

Checking Addition. To make sure that your problems are correct, you should learn to check the answers. This is important, especially in practical work. A good way to prove an answer in addition is to put down **below** the numbers included in the addition the sum of each column of figures separately arranged and added, as shown. Be sure to set each total one place to the left of the preceding one. (See Check B.)

Example	Check B	Check C
846	846	1208 check
142	142	846
16	16	142
195	195	16
9	9	195
1208 Ans.	28	9
	18	1208 check
	10	
	1208 check	

It saves time to use the process shown in Check B where long columns of figures are to be added. The sum of the first right-hand column is 28. The sum of the second column is 18. The sum of the third column is 10.

Another method of checking addition is first to add upward and then check by adding downward. If the two sums agree, the work is probably correct. Adding downward changes the order of the figures, therefore any error in the first addition would most likely be found in the second. Arrange as in Check C.

It is not difficult to add numbers when you have been told to find the sum. Sometimes you run across a problem in which you have trouble deciding on the process to use to solve it.

ILLUSTRATIVE EXAMPLES

1. How many bushels (the abbreviation bu. is generally **used**) are there in four bins with the following amounts, respectively: **4 bu.,** 52 bu., 268 bu., and 1873 bu.?

Solution

Instruction

Write the numbers one below the other with the units of each in the right-hand column, the tens in the next column to the left, the hundreds in the next, and the thousands in the next, just as shown. The plus (+) sign is not necessary when the numbers are so written and should not be used.

Add the figures in the units column first and set the total (17) below as shown. Now add the tens column including the 1 ten from adding the units. This gives 19, which is set below as shown. Add the hundreds column including the 1 hundred from adding the tens. This gives 11, set down as shown. Now add the thousands column including the 1 thousand from adding the hundreds and set down as shown. Your result is scattered, so set all right-hand figures in one line. 2197 is the sum of the number of bushels in the four bins. You will notice that when the sum of any column is more than 9, the tens figure is added to the next column to the left. This figure, 1 (in each case here) can be added without setting down as shown after you get used to adding as shown at the right.

In this case the 1 is added on without setting it down separately. By adding the 1 on when starting to add the next column, you do not need to remember it.

Operation

	Thousands	Hundreds	Tens	Units
				4
			5	2
		2	6	8
	1	8	7	3
			1	7
		1	9	
	1	1		
	2			
Total	2	1	9	7

```
   4
  52
 268
1873
----
2197
```

2. Five farmers sold their corn at one time, they had the following amounts: 1287 bu., 2193 bu., 4080 bu., 3923 bu., and 987 bu. How much was delivered to the elevator?

Solution

Instruction

Set the figures down in the same manner as before. In adding the units column you get 20, so set down the cipher in the units column and add the 2 to the tens column. Adding the tens column plus the above 2 equals 37. Set the seven in the tens column and add the 3 to the next column, which is hundreds. Adding the hundreds with the 3 additional gives 24. Set the 4 in the hundreds column and add the 2 to the next higher place, which is thousands. Adding the thousands plus the 2 gives 12. Put the two in the thousands place and carry the 1 to the next column. Place the one in the ten thousands place as there is nothing to add. The final sum is twelve thousand four hundred seventy.

Operation

```
 1287 bu.
 2193 bu.
 4080 bu.
 3923 bu.
  987 bu.
---------
12470 bu.
```

3. An electric light plant, which furnished current for 200 16-candlepower lamps, cost as follows: 16-horsepower engine 350 dollars; dynamo 275 dollars; driving belt 50 dollars; installation 35 dollars. What was the total cost?

Solution

Instruction

It will readily be seen after you read this problem several times that 200 plays no part in its solution. The cost will be found easily by simple addition, as shown below.

Operation

16-horsepower engine..........	$350	
Dynamo.....................	275	
Driving belt..................	50	
Installation..................	35	
Total cost................	$710	Ans.

4. A surveying party works for six weeks. The first week they survey 151 miles; the second week they survey 111 miles; the third week they survey 162 miles; the fourth week they survey 159 miles; the fifth week they survey 96 miles; and the sixth week they survey 48 miles. How many miles in all did they survey?

Solution

First week...............	151 miles	
Second week.............	111 miles	
Third week..............	162 miles	
Fourth week.............	159 miles	
Fifth week...............	96 miles	
Sixth week...............	48 miles	
	727 miles.	Ans.

5. When purchasing a 1,990 dollar automobile, the following extras were bought: 1 tire $50, 1 bumper $17, 1 spotlight $25, 1 radiator ornament $10. What was the total cost of the car?

Solution

Cost of automobile...........	$1,990	
Cost of 1 tire...............	50	
Cost of 1 bumper............	17	
Cost of 1 spotlight...........	25	
Cost of 1 radiator ornament...	10	
Total cost of car........	$2,092	Ans.

PRACTICE PROBLEMS

1. Find the sum of 56+49+17+36+21. Ans. 179
2. Find the sum of 467+536+84+705. Ans. 1,792
3. Find the sum of 8,950+15,765+7,732. Ans. 32,447

Lesson 3

For Step 1, recall that subtraction is taking away. If you have 24 dollars and 7 dollars are taken away, you have left 17 dollars. This is the process of subtraction. For Step 2, learn the method of subtracting numbers as explained in this lesson. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

SUBTRACTION

Subtraction is the opposite of addition. In addition you combine two numbers while in subtraction you take one number away from another. As in addition the quantities or numbers to be subtracted must be of the same kind and the same units.

There are three ways to define subtraction, the first two being most generally used.

(1) Subtraction is the process of finding the difference between two numbers.

(2) Subtraction is the process of taking one number from another.

(3) Subtraction is the process of finding what number must be added to a number to equal a given number.

The number from which you subtract is called the **minuend.**

The number to be subtracted is called the **subtrahend.**

The result or answer is the **difference** or **remainder.**

The sign − (read **minus**) indicates **subtraction.**

The Illustrative Examples will illustrate some of the difficulties met with in subtraction.

ILLUSTRATIVE EXAMPLES

1. If you take 34 apples away from 76 apples, how many apples have you left?

Solution

Instruction	*Operation*
Write the numbers, placing the minuend above the subtrahend.	76 minuend
Begin at the right-hand end and take 4 units from 6 units. This of course leaves 2 units. Now subtract the figures in the tens column. 3 from 7 = 4	34 subtrahend
In reality of course this is 30 from 70 = 40, but this is understood when the 4 is set in the tens place.	42 remainder

2. Subtract 364 from 942.

Solution

Instruction

Remember that 1 ten = 10 units
1 hundred = 10 tens
1 thousand = 10 hundreds

Write the numbers, placing the minuend above the subtrahend.

Operation

Step 1

4 cannot be taken from 2, so we take one ten from the tens place in the minuend and add it to the 2 units. Since one ten = 10 units, we now have 10+2, or 12 units. Subtract 4 units from 12 units, leaving 8 units. Since we took one ten from the tens place, we have only 3 tens left there.

Step 1

942 minuend
364 subtrahend

93(12)
36 4
8

Step 2

6 cannot be taken from 3, so we take one hundred from the hundreds place and add it to the 3 tens. One hundred = 10 tens, so we have 10+3, or 13 tens. Subtract 6 tens from 13 tens, leaving 7 tens. We took one hundred from the 9 hundreds, so have only 8 hundreds left. Subtract 3 hundreds from 8 hundreds leaving 5 hundreds. Our remainder is 5 hundreds, 7 tens, 8 units, and is read five hundred, seventy-eight.

Step 2

8(13)(12)
3 6 4
5 7 8

942
364
578 remainder

3. Subtract 3908 from 6505.

Solution

Step 1 *Instruction*

Write the numbers, putting minuend above the subtrahend. 8 cannot be taken from 5. If we now try to take one ten from the tens place (as we did in Problem 2), we find that there are no tens in the tens place.

Operation

Step 1

6505 minuend
3908 subtrahend

Step 2

So what we do is to take one hundred from the hundreds place and add it to the 0 tens. Since one hundred = 10 tens, we have 10+0, or 10 tens. (Note very carefully that there are now 10 tens in the tens place.)

Step 2

64(10)5
39 0 8

Step 3

Now take one ten from 10 tens and add it to the 5 units. Since one ten = 10 units, we have 10+5, or 15 units. Subtract 8 units from 15 units leaving 7 units.

Step 3

$$\begin{array}{r} 649(15) \\ 390\ \ 8 \\ \hline 7 \end{array}$$

Step 4

Since we took one ten from the 10 tens, there are only 9 tens left. Subtract 0 tens from 9 tens, leaving 9 tens. You will remember that we took one hundred from the 5 hundreds, so there are only 4 hundreds left.

Step 4

$$\begin{array}{r} 649(15) \\ 390\ \ 8 \\ \hline 9\ \ 7 \end{array}$$

Step 5

9 cannot be taken from 4, so take one thousand from the 6 thousands and add it to the 4 hundreds. One thousand = 10 hundreds, so we have 10+4, or 14 hundreds. Subtract 9 hundreds from 14 hundreds, leaving 5 hundreds. We took one thousand from the 6 thousands, so there are 5 thousands left. Subtract 3 thousands from 5 thousands leaving 2 thousands. Our remainder is 2 thousands, 5 hundreds, 9 tens, 7 units, and is read two thousand, five hundred ninety-seven.

Step 5

$$\begin{array}{r} 5(14)9(15) \\ 3\ \ 9\ 0\ \ 8 \\ \hline 2\ \ 5\ 9\ \ 7 \end{array}$$

$$\begin{array}{r} 6505 \\ 3908 \\ \hline 2597 \end{array}$$

4. Subtract 26 from 1000.

Solution

Instruction / *Operation*

Step 1

6 cannot be taken from 0. There are no tens and no hundreds to take anything from, so we have to go all the way back to the thousands place, and take one thousand from there (leaving 0 thousands). One thousand = 10 hundreds, so we have 10+0 or 10 hundreds.

Step 1

$$\begin{array}{r} 1000 \\ 26 \\ \hline 0(10)00 \\ 26 \end{array}$$

Step 2

Take one of these hundreds (leaving 9 hundreds). One hundred = 10 tens, so we have 10+0 or 10 tens.

Step 2

$$\begin{array}{r} 09(10)0 \\ 2\ 6 \end{array}$$

Step 3

Take one of these tens (leaving 9 tens). One ten = 10 units, so we have 10+0 or 10 units. Our problem now really is

$$\begin{array}{r} 099(10) \\ 2\ \ 6 \\ \hline \end{array}$$

leaving 097 4 or nine hundred seventy-four, as remainder.

Step 3

$$\begin{array}{r} 099(10) \\ 2\ \ 6 \\ \hline 097\ \ 4 \end{array}$$

$$\begin{array}{r} 1000 \\ 26 \\ \hline 974 \end{array}$$

5. Subtract each number below from 1000 and keep track of the time it takes.

1.	225	5.	725
2.	314	6.	328
3.	216	7.	900
4.	499	8.	831

Try again, timing yourself to see if you can beat your first record. To check your answers add the remainder to the subtrahend. The result should be the same as the minuend.

PRACTICE PROBLEMS

1. A coal shed contains 8579 tons. 3243 tons are taken from it. It then receives 4112 tons more. After that 1602 tons are taken out of it. How many tons remain? Ans. 7846

2. A tank has 1200 gallons of water in it. It loses 64 gallons by leakage and 32 gallons are pumped into it. How many gallons are now in the tank? Ans. 1168

3. A family took a week-end trip. The speedometer read 8090 miles before starting. The speedometer read 9428 miles when the family reached home. How far did they travel? (In order to find how far they traveled you deduct the meter reading at the start from the final reading.) Ans. 1338 miles, as 8090 miles had been traveled in previous trips.

Lesson 4

For Step 1, recall the parable of the loaves of bread, in which a few loaves were multiplied into many, that is, the few loaves became many times the original number. Thus, multiplication is taking a quantity a number of times and making a larger quantity. For Step 2, learn the method of multiplying quantities as explained in this lesson. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

MULTIPLICATION

Multiplication is a shortcut for addition. It is the process of taking one number as many times as there are units in another. Thus, 3 multiplied by 2 is the same as 3 taken 2 times or $3+3=6$.

TABLE II

$1\times2=2$	$1\times3=3$	$1\times4=4$
$2\times2=4$	$2\times3=6$	$2\times4=8$
$3\times2=6$	$3\times3=9$	$3\times4=12$
$4\times2=8$	$4\times3=12$	$4\times4=16$
$5\times2=10$	$5\times3=15$	$5\times4=20$
$6\times2=12$	$6\times3=18$	$6\times4=24$
$7\times2=14$	$7\times3=21$	$7\times4=28$
$8\times2=16$	$8\times3=24$	$8\times4=32$
$9\times2=18$	$9\times3=27$	$9\times4=36$
$10\times2=20$	$10\times3=30$	$10\times4=40$
$11\times2=22$	$11\times3=33$	$11\times4=44$
$12\times2=24$	$12\times3=36$	$12\times4=48$
$1\times6=6$	$1\times7=7$	$1\times8=8$
$2\times6=12$	$2\times7=14$	$2\times8=16$
$3\times6=18$	$3\times7=21$	$3\times8=24$
$4\times6=24$	$4\times7=28$	$4\times8=32$
$5\times6=30$	$5\times7=35$	$5\times8=40$
$6\times6=36$	$6\times7=42$	$6\times8=48$
$7\times6=42$	$7\times7=49$	$7\times8=56$
$8\times6=48$	$8\times7=56$	$8\times8=64$
$9\times6=54$	$9\times7=63$	$9\times8=72$
$10\times6=60$	$10\times7=70$	$10\times8=80$
$11\times6=66$	$11\times7=77$	$11\times8=88$
$12\times6=72$	$12\times7=84$	$12\times8=96$
$1\times10=10$	$1\times11=11$	$1\times12=12$
$2\times10=20$	$2\times11=22$	$2\times12=24$
$3\times10=30$	$3\times11=33$	$3\times12=36$
$4\times10=40$	$4\times11=44$	$4\times12=48$
$5\times10=50$	$5\times11=55$	$5\times12=60$
$6\times10=60$	$6\times11=66$	$6\times12=72$
$7\times10=70$	$7\times11=77$	$7\times12=84$
$8\times10=80$	$8\times11=88$	$8\times12=96$
$9\times10=90$	$9\times11=99$	$9\times12=108$
$10\times10=100$	$10\times11=110$	$10\times12=120$
$11\times10=110$	$11\times11=121$	$11\times12=132$
$12\times10=120$	$12\times11=132$	$12\times12=144$

Study the multiplication tables up to 12×12, in Table II, until you know them by heart. Each number from 2 to 12 is taken separately so that you can learn its products with numbers from 1 to 12. Begin with the lower numbers first and gradually increase until you have all of them thoroughly learned.

In multiplication there are three terms used, the multiplicand, the multiplier, and the product.

The number multiplied is called the **multiplicand.**

The number by which you multiply is called the **multiplier.**

The result of the multiplication is called the **product.**

The sign × (read **multiplied by**) means **multiplication.** Thus 6×5 is read 6 multiplied by 5.

A number **associated with** or **applied to** a particular object is called a **concrete number;** as 4 eggs, 5 ounces, 10 yards.

A number used **without reference** to a particular object is called an **abstract number;** as 7, 3, 10.

The **multiplier** is always thought of as an **abstract number.**

The multiplicand may be either concrete or abstract. When it is concrete, the product takes the same name as the multiplicand.

6 pounds	multiplicand
4	multiplier
24 pounds	product

This is the same as $6+6+6+6=24$

If both numbers of a multiplication problem are abstract, either one may be used as the multiplier. For example, $4 \times 7=28$ is the same as $7 \times 4=28$

In case one number is larger than the other, the larger number is usually used as the multiplicand.

Checking Multiplication. One way to check your multiplication is to set down the product for each two figures as in Method (a) in Illustrative Example 1.

Another way is to use the multiplier as the multiplicand and the multiplicand as the multiplier. The same product results as shown. Different products give a chance to find an error.

47	multiplicand
346	multiplier
282	
188	
141	
16262	product

ILLUSTRATIVE EXAMPLES

1. Multiply 346 by 47

Solution

Instruction

This is shown in detail, so that you can really see what is done.
Note carefully in what **places** the different products are put.

Operation

Method (a)

1	346	multiplicand
2	47	multiplier
3	42	$=7\times6$
4	28	$=7\times4$
5	21	$=7\times3$
6	24	$=4\times6$
7	16	$=4\times4$
8	12	$=4\times3$
	16262	= product

This time lines **3**, **4**, and **5** of (a) are added as the work is done. Here only 2 of 6×7 is set down and the **4** is added to 7×4, giving 32. The 2 is set down in the tens place. The **3** is added to 3×7, giving 24, which is set down as shown in line **3**. Line **4** is obtained the same way as line **3**. In (a) line **6**, the 2 is added to line **7** and the 1 in line **7** is added to 12 of line **8**, thus giving line 4 in (b). Adding lines **3** and **4** in (b) gives line **5**. In other words lines **3**, **4**, and **5** in (a) equal line **3** in (b) and lines **6**, **7**, and **8** in (a) equal line **4** in (b).

Method (b)

1	346	multiplicand
2	47	multiplier
3	2422	
4	1384	
5	16262	product

Method (b) is used in practice.

2. Multiply 2,928 by 364

```
   2928
    364
  11712
 17568
 8784
1065792
```

Multiply 5,698 by 792

```
   5698
    792
  11396
 51282
39886
4512816
```

You know that zero has no value, therefore, $0\times0=0$; $0\times8=0$; $942\times0=0$; $a\times0=0$. When there is a zero in the multiplier, you multiply by only those numbers which have value, being sure to place the right-hand figure of each separate product under the figure used to find it.

4. Multiply 432 by 103

$$\begin{array}{r} 432 \\ 103 \\ \hline 1296 \\ 432 \\ \hline 44496 \end{array}$$

5. Multiply 13,456 by 2,004

$$\begin{array}{r} 13456 \\ 2004 \\ \hline 53824 \\ 26912 \\ \hline 26965824 \end{array}$$

To multiply a number by 10, annex 0.	$38 \times 10 = 380$
To multiply a number by 100, annex 00.	$75 \times 100 = 7{,}500$
To multiply a number by 1000, annex 000.	$89 \times 1000 = 89{,}000$

6. Multiply 270 by 2,000

Solution

Instruction	*Operation*
Multiplying 27 by 2 gives 54. To 54 annex as many ciphers as there are in both the multiplicand and multiplier together, which is 4 here, and you have 540,000. (Annex means to place after.)	$\begin{array}{r} 270 \\ 2000 \\ \hline 540000 \end{array}$

A short way to multiply by 25 is to annex two ciphers and divide by 4. This is due to the fact that $100 \div 4 = 25$.

7. Multiply 78 by 25 $\quad 4\,\underline{)\,7800}$ $\quad 1950$

To multiply by 5, multiply by 10 and divide by 2; or annex one cipher and divide by 2. This is due to the fact that $10 \div 2 = 5$.

8. Multiply 420 by 5 $\quad 2\,\underline{)\,4200}$ $\quad 2100$

9. Multiply 875 by 5 $\quad 2\,\underline{)\,8750}$ $\quad 4375$

TABLE III

Multiplication Table

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168	175
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	168	176	184	192	200
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	189	198	207	216	225
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	250
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264	275
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288	300
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312	325
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	294	308	322	336	350
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	360	375
16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	336	352	368	384	400
17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340	357	374	391	408	425
18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360	378	396	414	432	450
19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380	399	418	437	456	475
20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400	420	440	460	480	500
21	42	63	84	105	126	147	168	189	210	231	252	273	294	315	336	357	378	399	420	441	462	483	504	525
22	44	66	88	110	132	154	176	198	220	242	264	286	308	330	352	374	396	418	440	462	484	506	528	550
23	46	69	92	115	138	161	184	207	230	253	276	299	322	345	358	391	414	437	460	483	506	529	552	575
24	48	72	96	120	144	168	192	216	240	264	288	312	336	360	384	408	432	456	480	504	528	552	576	600
25	50	75	100	125	150	175	200	225	250	275	300	325	350	375	400	425	450	475	500	525	550	575	600	625
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

Table III gives the product of any two numbers up to 25×25. To use this table, find one of the two numbers in the upper row and the other in the left-hand column. The product of the two will be found at the intersection of two imaginary lines drawn parallel to the heavy lines shown in the table at 12×12 and 20×20. For example, under the 9 and opposite the 8 is found the product 72, that is, 9×8=72.

PRACTICE PROBLEMS

1. If an airplane averages 97 miles an hour, how many miles will it travel in 16 hours? Ans. 1552 miles.

2. A man owns a sugar beet farm containing 1753 acres. How many tons of beets does he raise if the average yield is 12 tons per acre? Ans. 21,036 tons.

3. Subtract 175 from 5208 and multiply the difference by 97. Ans. 488,201.

4. What would be the value of 867 shares of railroad stock at \$97 a share? Ans. \$84,099.

5. If one mile of railroad requires 116 tons of iron worth \$65 a ton, what will be the cost of sufficient iron to construct a road 128 miles in length? Ans. \$965,120.

6. A mechanic receives \$56 for 9 days work, and spends \$3 a day for the whole time. How much has he left? Ans. \$29.

7. A man bought 32 loads of wheat, each load containing 50 bushels, at \$2 a bushel. What did the wheat cost? Ans. \$3200.

Lesson 5

For Step 1, recall that in your daily life you have had occasion to divide certain objects among other people. When a child, you divided marbles; possibly you broke an apple in two and gave half to a friend. These are acts of division. For Step 2, learn the method of dividing numbers as explained in this lesson. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

DIVISION

Division is the process of finding how many times one number contains another. It is the reverse of multiplication. $7 \times 8 = 56$; 56 divided by $8 = 7$; 56 divided by $7 = 8$.

In division the terms used are dividend, divisor, quotient, and remainder.

The number to be divided is called the **dividend.**

The number by which you divide is called the **divisor.**

The result of division is called the **quotient.**

The part of the dividend left over when the quotient is not exact is called the **remainder.**

The sign $\div$ (read **divided by**) means **division.** $6 \div 3 = 2$, or $\frac{6}{3} = 2$. The line between means division.

In the process of division two methods are used: **short division and long division.** Short division is generally used when the divisor has only one figure other than ciphers.

ILLUSTRATIVE EXAMPLES

1. Divide 720 by 5—short division

Solution

Instruction

In the first place we find mentally the number which, when multiplied by the divisor, gives a product equal to the first number in the dividend, or if not equal, as near equal as possible, but not larger than the first number mentioned. In this problem, 5 is the divisor and 7 is the first number in question; then the number is 1, because $1\times5=5$ and $2\times5=10$. As 10 is larger than seven, we must take 1 and place it as the first number in the quotient.

Multiply mentally 1 by the divisor 5 and subtract the product from the first number or 7. The difference is 2. Place this number 2 mentally in front of the second number in the quantity or dividend and form mentally the number 22. Find the number that multiplied by the divisor 5 will give 22 or near it. The number is 4, because $5\times4=20$. Place the 4 next to the 1 in the quotient.

Multiply 4 by the divisor 5 and subtract the product from the 22. The difference is 2. Place 2 mentally in front of the next number of the dividend and you will get 20. Find the number that multiplied by the divisor 5 will give 20 or near it. The number is 4, because $5\times4=20$. Place the 4 as the last figure in the quotient. The quotient or answer is 144.

To check this result, just reverse the process and multiply 144 by 5. If the division is correct, the product will be equal to the dividend. In this problem, 5×144 is equal to 720 or the dividend, and the work checks.

Operation

$$\text{divisor } 5\overline{)720} \text{ dividend}$$
$$144 \text{ quotient}$$

2. Divide 6785 by 21.

Solution

Instruction

Step 1

We cannot readily tell how many times 21 is contained in 67, so to find the first figure of quotient, divide the first figure of the dividend (6) by first figure of divisor (2). $6\div2=3$. Put down 3 as first figure of quotient. Multiply entire divisor, 21, by 3. $21\times3=63$. Put 63 under 67 and subtract. The remainder is 4.

Operation

Step 1

$$\begin{array}{rl} 3 & \text{quotient} \\ \text{divisor } 21\overline{)6785} & \\ \underline{63} & \\ 4 & \end{array}$$

Step 2

Annex the next figure (8) of the dividend. Then our new dividend is 48. As before, divide the first figure of this dividend (4) by first figure of divisor (2). $4 \div 2 = 2$. Put down 2 as second figure of quotient. Multiply entire divisor, 21, by 2. $21 \times 2 = 42$. Put 42 under 48 and subtract. The remainder is 6.

Step 2

$$\begin{array}{r} 32 \\ 21\,\overline{)\,6785} \\ \underline{63} \\ 48 \\ \underline{42} \\ 6 \end{array}$$

Step 3

Annex the next figure (5) of original dividend. Our new dividend is 65. Again divide first figure (6) of dividend by first figure (2) of divisor. $6 \div 2 = 3$. Put down 3 as third figure of quotient. Multiply entire divisor, 21, by 3. $21 \times 3 = 63$. Put 63 under 65 and subtract. The remainder is 2. Since there are no further figures in original dividend, our work is complete.

Step 3

$$\begin{array}{rl} 323 & \text{quotient} \\ \text{divisor } 21\,\overline{)\,6785} & \text{dividend} \\ \underline{63} & \\ 48 & \\ \underline{42} & \\ 65 & \\ \underline{63} & \\ 2 & \end{array}$$

3. Divide 8936 by 34. (Follow same procedure as was used in Problem 2.)

Solution

Instruction — *Operation*

Step 1

To find first figure of quotient, divide 8 by 3. $8 \div 3 = 2$. Put the 2 down as first figure of quotient. Multiply 34 by 2. $34 \times 2 = 68$. Put 68 under 89 and subtract. The remainder is 21.

Step 1

$$\begin{array}{r} 34\,\overline{)\,8936} \\ \underline{68} \\ 21 \end{array}$$

Step 2

Annex the next figure (3) of the dividend. Our new dividend is 213. There are three figures in this dividend and only two in the divisor, so divide the first two figures (21) of dividend by first figure (3) of divisor, $21 \div 3 = 7$. Put 7 as the second figure of quotient. Multiply 34 by 7. $34 \times 7 = 238$. Put 238 under 213. You will notice that 238 is a larger number than 213, so we cannot subtract. Therefore, we must put a smaller number than 7 in the quotient. Try 6. Multiply 34 by 6. $34 \times 6 = 204$. Put 204 under 213 and subtract. The remainder is 9.

Step 2

$$\begin{array}{r} 26 \\ 34\,\overline{)\,8936} \\ \underline{68} \\ 213 \\ \underline{204} \\ 9 \end{array}$$

Step 3

Annex the next figure (6) of original dividend. Our new dividend is 96. To find third figure of quotient divide 9 by 3. $9 \div 3 = 3$. Put 3 down as third figure in quotient. Multiply 34 by 3. $34 \times 3 = 102$. Put 102 under 96. But we see that 102 is larger than 96, and we cannot subtract. So we must put a smaller number than 3 in quotient. Try 2. Multiply 34 by 2. $34 \times 2 = 68$. Put 68 under 96 and subtract. The remainder is 28. There are no further figures in original dividend, so our work is complete. In solving division problems you will have to note carefully that the number you get after multiplying is smaller than the number you are to subtract it from. If it is larger, you must try a smaller number in the quotient.

Step 3

$$\begin{array}{r} 262 \\ 34\,)\overline{8936} \\ \underline{68} \\ 213 \\ \underline{204} \\ 96 \\ \underline{68} \\ 28 \end{array}$$

4. Divide 104270 by 13.

Solution

Instruction / *Operation*

Step 1

Divide first two figures (10) of dividend by first figure (1) of divisor. $10 \div 1 = 10$. But 10 will be found to be too large. If we try 9, we shall find that it also is too large, so put down 8 as the first figure of quotient. Multiply 13 by 8. $13 \times 8 = 104$. Put 104 under 104 and subtract. The remainder is 0.

Step 1

$$\begin{array}{r} 8 \\ 13\,)\overline{104270} \\ \underline{104} \\ 0 \end{array}$$

Step 2

Annex the next figure (2) of dividend. Now, divide first figure (0) of dividend by first figure (1) of divisor $0 \div 1 = 0$. Put 0 as second figure of quotient. Multiply 13 by 0. $13 \times 0 = 0$. Subtract 0 from 2, leaving 2. (We do not write down these operations, as multiplying by 0, and subtracting 0, have no effect on the figuring.)

Step 2

$$\begin{array}{r} 80 \\ 13\,)\overline{104270} \\ \underline{104} \\ 02 \end{array}$$

Step 3

Annex next figure (7) of original dividend. Divide 2 by 1. $2 \div 1 = 2$. Put down 2 as third figure in quotient. Multiply 13 by 2. $13 \times 2 = 26$. Put 26 under 27 and subtract. The remainder is 1.

Step 3

$$\begin{array}{r} 802 \\ 13\,)\overline{104270} \\ \underline{104} \\ 027 \\ \underline{26} \\ 1 \end{array}$$

Step 4

Annex next figure (0) of original dividend. Divide 1 by 1. $1 \div 1 = 1$. If we multiply 13 by 1 we find that we have too large a number to subtract from 10. So we must put a smaller number than 1 in quotient. Put 0 as the fourth figure in quotient. Multiply 13 by 0. 13×0 is 0. Subtract 0 from 10, leaving 10 as remainder.

Step 4

$$\begin{array}{r} 8020 \\ 13\overline{)\,104270} \\ \underline{104} \\ 027 \\ \underline{26} \\ 10 \end{array}$$

5. Divide 151325 by 173

Operation

$$\begin{array}{rl} 874 & \text{quotient} \\ \text{divisor } 173\overline{)\,151325} & \text{dividend} \\ 1384 & \\ 1292 & \\ \underline{1211} & \\ 815 & \\ \underline{692} & \\ 123 & \text{remainder} \end{array}$$

To check, multiply the quotient by the divisor and add the remainder to the product.

Check

$$\begin{array}{r} 874 \\ \underline{173} \\ 2622 \\ 6118 \\ \underline{874} \\ 151202 \\ \underline{123} \\ 151325 \end{array}$$

To divide a number by 10, 100, etc., the shortest way is to cut off as many places from the right of the dividend as there are ciphers in the divisor.

$80 \div 10$, cut off cipher, $= 8\cancel{0}$ 8 Ans.
$10{,}000 \div 100$, cut off two ciphers, $= 100\cancel{0}\cancel{0}$ 100 Ans.
$30{,}000 \div 1000$, cut off three ciphers, $= 30\cancel{0}\cancel{0}\cancel{0}$ 30 Ans.

6. Divide 1218 by 100

$$\begin{array}{rl} 1/00\,)\,\underline{12/18} & \\ 12/18 & \text{remainder} \end{array}$$

Proof

$$\begin{array}{rl} 12 \times 100 = 1200 & \\ \underline{18} & \text{remainder} \\ 1218 & \end{array}$$

The two figures cut off are the remainder and the two at the left are the quotient. Thus $1218 \div 100 = 12$ and a remainder of 18. The result is written $12\frac{18}{100}$

7. There are 2000 pounds in a ton. If a coal dealer sells 36,000 pounds of coal in a day, how many tons does he sell? How many tons does he sell if he sells 36,357 pounds? 37,407 pounds? You

are to find how many times 2000 is contained in each of these numbers, therefore, you divide as follows:

2/000) 36/000	2/000) 36/357	2/000) 37/407
18	18/357	18/1407
18 tons Ans.	$18\frac{357}{2000}$ tons Ans.	$18\frac{1407}{2000}$ tons Ans.

8. Divide 8700 by 25

$$\begin{array}{r} 8700 \\ 4 \\ \hline 348\cancel{0}\cancel{0} \end{array} \quad \text{or } 348 \quad \text{Ans.}$$

To divide by 25, multiply the dividend by 4 and divide the product by 100, cutting off two figures from the right.

9. An afternoon newspaper in a certain city sold copies of the paper during one week as follows: Monday, 9462 copies; Tuesday, 10,987 copies; Wednesday, 8455 copies; Thursday, 12,309 copies; Friday, 11,087 copies; Saturday, 15,410 copies. How many papers were sold during the six days? What was the average sale per day?

Solution

$$\begin{array}{r} 9462 \\ 10987 \\ 8455 \\ 12309 \\ 11087 \\ 15410 \\ \hline 67710 \end{array} \text{ copies sold in 6 days. Ans.}$$

$$6\,\overline{)\,67710}$$
11285 average number sold per day. Ans.

10. The cost of running a bus line between two cities during one month was as follows: Labor, \$328; gasoline, \$298; repairs, \$78; depreciation, \$175. The income from passenger fares during that time was \$1,147. Did the bus company gain or lose? How much?

Solution

$$\begin{array}{r} \$328 \\ 298 \\ 78 \\ 175 \\ \hline \$879 \end{array} \text{ total expense for running bus 1 mo.}$$

$$\begin{array}{r} \$1,147 \\ 879 \\ \hline \$268 \end{array} \begin{array}{l} \text{income for 1 mo.} \\ \\ \text{gain. Ans.} \end{array}$$

11. If an auto runs at the rate of 28 miles an hour, how many hours will it take it to go 1,148 miles?

Solution

$$\begin{array}{r l} 41 & \text{hours.} \quad \text{Ans.} \\ 28\overline{)1148} & \\ \underline{112} & \\ 28 & \\ \underline{28} & \end{array}$$

To check division, multiply the divisor by the answer (quotient). If you have made no error, this will give the original number, or dividend. In Example 11, $28 \times 41 = 1148$.

PRACTICE PROBLEMS

1. Divide 414 by 18 — Ans. 23
2. Divide 1,656 by 23 — Ans. 72
3. Divide 11,022 by 22 — Ans. 501
4. Divide 37,185 by 88 — Ans. $422\frac{49}{88}$
5. Divide 295,625 by 43 — Ans. 6875
6. If coal costs $11 per ton, how many tons can be bought for 165 dollars? 15 tons. Ans.
7. If a train runs at the rate of 50 miles per hour, how long will it take to make the trip from Chicago to Salt Lake City, a distance of 1,400 miles? 28 hrs. Ans.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on Page 28 (top folio).

If you miss more than two of the problems it means you should review Section 1 carefully.

Do not try this trial examination until you have worked every practice problem in the Section.

Do not start the final examination until you have completed this trial examination.

1. Show how to multiply 666 by 100 using the short method.

2. Show how to multiply 243 by 3200 using the short method.

3. Show how to multiply 460 by 25 using the short method.

4. A farmer owned three parcels of land containing 240, 160, and 640 acres. Divided equally among his 4 sons, how much was each son's share?

5. A garage bought 500 gallons of gasoline three times, 1250 gallons twice, and 1500 gallons four times. If all this gasoline was sold in 20 days, what was the average number of gallons sold per day?

6. A wire stretches from the top of one pole to the top of another. The top of one pole is 20 feet above the water surface of a lake, and the top of the other is 6 feet below water level. How much higher is one end of the wire than the other?

7. An electrician cut 93 feet, 70 feet, 124 feet, 316 feet, 42 feet, and 4 times 12 feet, off a reel of wire which contained 1240 feet of wire. How much wire was left on the reel?

8. If you were working for a salary of $35 a week and received a raise to $2444 a year, name the increase per week, counting 52 weeks in a year?

9. A contractor paid the wages of 6 laborers who worked 5 days for 8 hours a day at $1 per hour. What was the total money paid out?

10. If every citizen of a community of 600,000 people gave only 1¢ each day to support a community chest, how large a sum of money in dollars would that amount to in 365 days?

11. Each of 40,000 families in a city pays $2 a month for gas and $3 a month for electricity. In 12 months, how much money is spent by all these families for these two items combined?

12. A business agent had to use a taxi each day at an average daily expense of $2. He estimated that to buy and operate a car would cost an average amount of $900 a year. Allowing 300 business days per year, how much more would it cost him per day to use his own car?

13. A welfare organization gave out 600 meals each day during a period of 120 days. If it cost $1 for every 10 meals, how much did these meals cost?

14. A man bought a new car. He paid $500 in cash. He received $250 for his old car. He paid the balance in 12 monthly payments of $80 each. What was the amount paid for the new car?

15. If a car is driven five days at the rate of 300, 200, 270, 350, and 180 miles per day, what is the average number of miles driven per day?

EXAMINATION

1. Write in words:
501
7,808
13,020
734,507,000

2. Write in figures:
Twenty-seven
One hundred twenty-three
Three thousand, thirty-nine
Ten thousand, six
Two million, four thousand, one hundred twelve

3. (a) A book contains forty-two chapters. Number the following by use of Roman Notation: Chapter nineteen, Chapter thirty-one, Chapter forty.

(b) Write twenty thousand the shortest way in Roman numerals.

(c) Write the value of M in figures.

4. What is the meaning of *product* in mathematics? Give an example, naming the product.

5. Find the sums in the following problems, and write down the number of minutes you spent adding them:

(a)	12	(b)	234	(c)	1708
	25		106		2596
	33		753		4372
	47		631		3284
	54		842		6053
	66		925		9167
	78				
	99				

6. From 1203 subtract each of the following numbers in turn: 325, 309, 724, 1102, and 588. Write down the time it took to subtract all five numbers, then check the first answer and show your work.

7. (a) Multiply 4375 by 86 and show a process of checking.
(b) Multiply 187 by 2000, using short method.
(c) Multiply 4123 by 3002, using short method.

8. (a) Divide 136,647 by 723 and check your answer, showing your method of checking.
(b) Divide 25,600 by 80, using short method.

9. The distance from San Francisco to Ogden is 783 miles; Ogden to Cheyenne, 483 miles; Cheyenne to Omaha, 507 miles; Omaha to Chicago, 488 miles; Chicago to New York, 988 miles; how far is it from San Francisco to New York?

10. A truck loaded with coal weighed 14,025 pounds. When empty the truck weighed 4985 pounds. How many pounds of coal in the truck when it was loaded?

11. Denver is 5182 feet above sea level; Death Valley is 271 feet below sea level. How much higher is Denver than Death Valley?

12. A man sold his home for $8,764, his store for $9,145, his city property for $16,381, and received in payment farm land worth $45 per acre. How many acres were in the farm?

13. A man paid $21,600 for a farm and sold it for $24,375. What was his profit?

14. In a seven-story hotel with 22 rooms on each floor, 924 tungsten lamps are used. If each room contains the same number of lamps, how many are there in each?

15. On Monday morning Thomas Knox and family started on an automobile trip. The speedometer registered 2,025 miles before starting. On Monday night it registered 2,299 miles; Tuesday night, 2,558 miles; Wednesday night, 2,853 miles; Thursday night, 3,145 miles; Friday night, 3,419 miles. How many miles did they travel each day?

16. How many street cars, each carrying 66 persons, will it take to carry 1,122 persons?

17. A man has $2,660 with which to buy cattle. At $65 each, how many cows can he buy, and how much money will he have left?

18. The owner of a garage had 720 gallons of gasoline at the beginning of the month. During the month he bought 1,200 gallons and sold 1,078 gallons. At the end of the month how many gallons were left?

19. During January the manager of a moving picture house took in $1,935. During the month he paid $535 for pictures, $275 for rent, $470 for wages, and $245 for taxes. How much did he make?

20. A factory worker who received $27 per week was promoted to a position paying $145 per month. Counting 52 weeks as one year, how much more does he receive a year in his new position?

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. To multiply any number by 100 add two ciphers to the number. Thus 666 multiplied by 100 is 66600.

2. To multiply 243 by 3200, place the 3200 under the 243 as shown in the following:

$$\begin{array}{r} 243 \\ \underline{3200} \end{array}$$

The first step is to bring down the two ciphers and put them in the answer:

$$\begin{array}{r} 243 \\ \underline{3200} \\ 00 \end{array}$$

The next step is to multiply 243 by 2:

$$\begin{array}{r} 243 \\ \underline{3200} \\ 48600 \end{array}$$

Then multiply 243 by 3 and remember that the second product must have its first number under the 8:

$$\begin{array}{rl} 243 & \\ \underline{3200} & \\ 48600 & \\ \underline{729} & \\ 777600 & \text{Ans.} \end{array}$$

3. A short way to multiply any number by 25 is to add two ciphers to the number and divide the number by 4. Thus, adding two ciphers to 460 we have 46000. Then $46000 \div 4 = 11500$ Ans.

Proof

$$\begin{array}{r} 460 \\ \underline{25} \\ 2300 \\ \underline{920} \\ 11500 \end{array}$$

4. First find the total number of acres the farmer owned.

$$\begin{array}{rl} 240 & \\ 160 & \\ \underline{640} & \\ 1040 & \text{(total acres)} \end{array}$$

If four sons received an equal share, then each would receive 1040 divided by 4 or 260 acres.

$$\begin{array}{rl} 260 & \text{Ans.} \\ 4\overline{)1040} & \\ \\ 24 & \\ \underline{24} & \\ 0 & \end{array}$$

5. First find the total amount of gasoline purchased.

$$\begin{array}{r}500\\ \underline{3}\\ 1500\end{array}\qquad\qquad\begin{array}{r}1250\\ \underline{2}\\ 2500\end{array}\qquad\qquad\begin{array}{r}1500\\ \underline{4}\\ 6000\end{array}$$

This is done by multiplication. The garage bought 500 gallons 3 times, so $500 \times 3 = 1500$. Then 1250 gallons were bought twice, so $1250 \times 2 = 2500$. Finally, 1500 gallons were purchased at 4 different times, so $1500 \times 4 = 6000$.

Next add to find total gallons of gasoline purchase.

$$\begin{array}{rl}1500&\\ 2500&\\ 6000&\\ 10000&\text{(total gallons)}\end{array}$$

To find the average number of gallons sold per day for 20 days, divide 10000 by 20.

$$\begin{array}{r}500\\ 20\overline{)10000}\\ \underline{100}\\ 00\end{array}$$

Therefore 500 gallons was the average sale per day.

6. The top of one pole is 20 feet above the water surface. The top of the other pole is 6 feet below the water surface. Therefore the difference in height between the tops of the poles is $20+6=26$ feet. This can be understood easily by remembering that the distance from the top of one pole to the water surface is 6 feet and that from the water surface to the top of the other pole is an additional 20 feet. Thus one end of the wire is 26 feet higher than the other.

7. First add the various lengths which were cut off the reel to find the total amount taken off the reel.

$$\begin{array}{rl}93&\\ 70&\\ 124&\\ 316&\\ 42&\\ \underline{48}&\\ 693&\text{feet cut off}\end{array}$$

In this column of figures the 48 was obtained by multiplying 12 feet by 4. This was necessary because on 4 occasions a 12-foot length was cut off. To find how many feet of wire were left on the 1240-foot reel, subtract 693 from 1240

$$\begin{array}{rl}1240&\\ \underline{693}&\\ 547&\text{feet left on reel}\end{array}$$

8. Before we can find out how much more your weekly salary would be after your raise, we must find the new weekly salary. We can do this by dividing $2444 by 52.

$$
\begin{array}{r}
47 \\
52\overline{)2444} \\
\underline{208} \\
364 \\
\underline{364}
\end{array}
$$

Your new weekly salary would be $47. Then to find how much more the new weekly salary is than the old, subtract 35 from 47.

$$
\begin{array}{r}
\$47 \\
\underline{35} \\
\$12 \text{ Ans.}
\end{array}
$$

9. If each laborer makes $1 per hour, multiply $1 by 8 to find the daily pay for one man.

$$\$1 \times 8 = \$8$$

In 5 days a laborer would make $\$8 \times 5 = \$40$. There are 6 laborers. Therefore $\$40 \times 6 = \$240$, the total money paid them.

10. We know that there are 100 cents in a dollar. To find the number of dollars 600,000 people gave per day, divide 600,000 by 100.

$$
\begin{array}{r}
6000 \\
100\overline{)600000} \\
\underline{600} \\
000
\end{array}
$$

This can be divided by the short method as follows: as there are two ciphers in the divisor, two ciphers are cut off from the right of the dividend thus, 600,0$\cancel{0}\cancel{0}$. Thus the 600,000 people gave $6000 every day.

To find the yearly amount, multiply $6000 by 365.

$$
\begin{array}{r}
\$6000 \\
\underline{365} \\
30000 \\
36000 \\
\underline{18000} \\
\$2190000 \text{ Ans.}
\end{array}
$$

This multiplying could have been done by the short method.

$$
\begin{array}{r}
365 \\
\underline{6000} \\
2190000
\end{array}
$$

11. If each of 40,000 families spends $2 for gas and $3 for electricity, then $\$2 + \$3 = \$5$ per month is spent per family.

For 40,000 families the total monthly expenditure is 40,000 multiplied by $5.

$$
\begin{array}{r}
40000 \\
\underline{5} \\
\$200000 \text{ (total per month)}
\end{array}
$$

For 12 months the total expenditure will be $200,000 multiplied by 12. We will do this by the short method.

$$\begin{array}{r} 12 \\ \underline{200000} \\ \$2400000 \text{ Ans.} \end{array}$$

12. To find the average daily cost of operating a car of his own instead of using a taxi, divide 900 by 300.

$$\begin{array}{r} 3 \\ 300\overline{)900} \\ \underline{900} \end{array}$$

This shows the daily average cost would be $3. Therefore it would cost the business agent $3 − 2 = $1 more per day to use his own car.

13. Every 10 meals cost $1. To find the number of dollars it costs to give out 600 meals in one day, divide 600 by 10.

$$\begin{array}{r} 60 \\ 10\overline{)600} \\ \underline{60} \\ 0 \end{array}$$

Thus for one day it costs $60 to give out 600 meals. Then for 120 days the total cost is 120 multiplied by $60.

$$\begin{array}{r} 120 \\ \underline{60} \\ \$7200 \text{ Ans.} \end{array}$$

14. First find the total amount of money paid by monthly payments. This is done by multiplying $80 by 12.

$$\begin{array}{r} \$80 \\ \underline{12} \\ 160 \\ 80 \end{array}$$

To this $960 add the cash payment of $500 and the $250 amount received for the old car.

$$\begin{array}{r} 500 \\ \underline{250} \\ \$1710 \text{ (price of car)} \end{array}$$

15. Add the various distances to find the total distance driven in 5 days.

$$\begin{array}{r} 300 \\ 200 \\ 270 \\ 350 \\ \underline{180} \\ 1300 \text{ (total miles)} \end{array}$$

To find the average number of miles driven per day, divide 1300 by 5.

$$\begin{array}{r} 260 \\ 5\overline{)1300} \\ \underline{10} \\ 30 \\ \underline{30} \\ 0 \end{array}$$

The answer is 260 miles.

PRACTICAL MATHEMATICS

Section 2

FACTORING AND CANCELLATION

Lesson 1

For Step 1, recall what you learned in Section 1 about multiplication and division. Be sure you have a good understanding of these two items because Section 2 is based, to a great extent, on such operations. For Step 2, study the definitions until you fully understand them. For Step 3, study the Illustrative Examples. For Step 4, work the Practice Problems.

Definitions

In Section 1 you learned definitions of such words as Period, Sum, Minuend, Subtrahend, Multiplicand, Quotient, Remainder, etc. You also learned how these words were used in addition, subtraction, multiplication, and division. In this section you will learn some new words which are equally important.

Whole Numbers. For the present consider any and all numbers, from 1 to a million or more, as **whole numbers.** Thus you can see that **numbers** which are used in Section 1 are all whole numbers.

Integer. The word **integer** means exactly the same as **whole number.** In mathematics either word may be used to indicate such numbers as you became familiar with in Section 1.

Odd Numbers. In your study of division, in Section 1, you learned that a **remainder** is that part of a dividend left over when the quotient is not exact. In other words if we divide 13 by 2 we obtain a quotient of 6 and have a remainder of 1. In this case we say that the 2 did not divide into 13 an exact number of times. If we divide 12 by 2, the quotient is 6 without any remainder. Here we say that 2 divides into 12 an exact number of times. *Thus when we say that one number divides into another number an exact number of times, we mean there is no remainder.*

An **odd number** is any number that cannot be divided by 2 an exact number of times.

A quick way to recognize an odd number is to remember that any number ending in 1, 3, 5, 7, or 9 cannot be divided an exact number of times by 2 and is, therefore, an odd number.

Even Numbers. An **even number** is any number that can be exactly divided by 2.

A quick way to recognize an even number is to remember that any number ending in 2, 4, 6, 8, or 0 can be exactly divided by 2.

You should understand that both odd and even numbers are also whole numbers and integers. Remember the definitions for whole numbers and integers and you will understand this.

Prime Numbers. A **prime number** is any number that can be divided an exact number of times *only* by itself and 1. Take the number 19 for example. Only the numbers 19 and 1 divide into 19 an exact number of times. Therefore it is a **prime** number. There are many prime numbers, a few of which are 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, etc.

Factors. In Section 1 you learned that if we multiply one number by another number the answer is called the product. For example, $2\times7=14$. Fourteen is the product. The 2 and 7 are called **factors.** Then we say that the factors of 14 are 2 and 7. In other words, the factors of a number are those numbers which, when multiplied together, give a product which is the number we have in mind.

Some numbers have more than two factors. Take the number 105 for example. If we multiply $3\times5\times7$ the product is 105. Here we say that 3, 5, and 7 are factors of 105. The number 105 also has the factors 3 and 35 because $3\times35=105$. Some numbers have more than two or three factors. Very large numbers can have a great number of factors and, like the number 105, may have different sets of factors. However, only one set can be used at one time. Remember that the factors of a number when multiplied together must give a product which is the number we have in mind.

When we talk of factors of a number, the factors may be prime numbers or ordinary numbers, or combinations of both.

Prime Factors. A **prime factor** is a factor consisting of a prime number. Sometimes it is necessary to find the prime factors of a given integer or whole number. In this case remember the definition of a prime number and select only such factors as are prime numbers.

The integer 1 is a prime number but it should never be used

as a factor. Also, the number for which factors are being found should never be used as a factor.

Integers Prime to Each Other. If we have two or more integers that cannot be divided an exact number of times by any other number, we say these integers are **prime to each other.** Take 7 and 11 for example. By trial we find that no other number will divide into either 7 or 11 without a remainder. Therefore the integers 7 and 11 are prime to each other. Both 7 and 11 are prime numbers themselves and are prime to each other as well.

Ordinary integers (not prime numbers) can also be prime to each other. For example, consider the integers 18 and 25. These integers are not prime numbers in themselves because each can be divided by one or more numbers in addition to itself and 1. The 18 can be divided by 6 and 9 an exact number of times, and the 25 by 5. But, there is no number that will divide into *both* 18 and 25 an exact number of times. Therefore we say that 18 and 25 are prime to each other.

Ordinary Factors. An **ordinary factor** is any number whether it is a prime factor or not. Thus 2, 3, 4, 5, 6, 7, 8, 9, 15, 18, etc., are all ordinary factors. Remember that ordinary factors can be any number, whether odd or even or prime. In fact, any number can be an ordinary factor.

Common Factors. If two or more ordinary factors divide an exact number of times into two or more integers such factors are called **common factors.** It will be seen that 2, 4, 7, and 14 all divide an exact number of times into the integers 28 and 84. Therefore the 2, 4, 7, and 14 are the common factors of 28 and 84. (This process is explained in Lesson 2.)

Highest Common Factor. After we have found all the common factors of two or more integers, we can easily select the largest of them.

In the preceding illustration, 14 is the largest of the common factors. The 14 is therefore the **highest common factor** because it is the largest factor which divides into 28 and 84 an exact number of times.

ILLUSTRATIVE EXAMPLES

1. Write five whole numbers.

Solution. We know from the definition that a whole number is any number that comes to mind. Therefore, 8, 15, 467, 31, and 88 are typical whole numbers.

2. Write five integers.

Solution. We know from the definition that an integer is the same as a whole number. Therefore 8, 15, 467, 31, and 88 are also integers. Or 29, 333, 919, 858, 8222 are also integers.

3. Write five odd numbers.

Solution. We know from the definition that an odd number is one that cannot be divided by 2 without leaving a remainder. Thus 31, 57, 11, 13, and 61 are all odd numbers.

4. Write five even numbers.

Solution. We know from the definition that an even number can be divided by 2 without leaving a remainder. Therefore 2, 10, 50, 34, 58 are even numbers.

5. Write five prime numbers.

Solution. We know from the definition that a prime number is a number which can only be divided an exact number of times by itself and 1. Therefore 53, 59, 61, and 67 are all prime numbers.

6. Write five numbers which could be prime factors.

Solution. A prime factor must be a prime number while any number can be a factor. This means then that we have to select five prime numbers. The numbers 2, 3, 5, 7, and 11 satisfy these conditions.

7. Write five numbers which could be common factors.

Solution. Any five numbers we can think of might be common factors, so 10, 12, 13, 17, and 50 are satisfactory.

8. If 2, 3, 4, 6, and 12 are the common factors of 36 and 48 which is the highest common factor?

Solution. The 12 is the highest common factor because it is the largest factor which will divide an exact number of times into both 36 and 48.

PRACTICE PROBLEMS

After you have worked all of the problems, compare your answers with the correct answers shown on page 18.

If you have one or more wrong answers, be sure to review the definitions and illustrative examples very carefully.

1. Write out all of the integers from 1 to 12.
2. Write the odd numbers which appear from 17 to 41.
3. Write out the even numbers which appear from 40 to 60.
4. Write out the prime numbers which appear from 13 to 43.

5. If two integers can be divided by the numbers 2, 8, 10, 12, and 14 an exact number of times, what are these numbers called?

6. If no one number will exactly divide into two integers, what do we say about these integers?

7. If a number is the product of two integers, what other name could we give to the two integers?

Lesson 2

For Step 1, recall all the definitions of Lesson 1. For Step 2, learn the methods of factoring. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

FACTORING

In Lesson 1 you learned that there are three different kinds of factors and that they are called **prime, ordinary**, and **common** factors. The methods of finding these three types of factors are different and are explained as follows.

To Find Prime Factors. To find the prime factors of any number, first divide it by the smallest prime factor, above 1, that will divide into it an exact number of times. Such a prime number can be found by trial and by always trying 2 first, then 3, 5, 7, 11, etc., as required. After the divisor is found and the quotient obtained, find the smallest prime number that will divide an exact number of times into the quotient. Continue this process until a quotient is obtained which is a prime number. When a prime number quotient is obtained, the factoring process is complete because a prime number cannot be divided by any number except itself and 1. The factors for a number include the several divisors and the last quotient. Never use 1 or the number being factored as divisors.

ILLUSTRATIVE EXAMPLES

1. Find the prime factors of 16.

Solution

$$\begin{array}{r} 2\underline{)16} \\ 2\underline{)\ 8} \\ 2\underline{)\ 4} \\ 2 \end{array}$$

Prime factors are 2, 2, 2, and 2.

The first step is to divide 16 by the smallest prime number (above 1) which will divide into it an exact number of times. First try 2. This divides 16 exactly 8 times, so it is satisfactory. The

next step is to find the smallest prime number that will divide into the quotient (8) an exact number of times. The prime factor 2 can be used again, leaving a quotient of 4. The third step is to divide the quotient (4) by the smallest prime number which will divide into it an exact number of times. Once again 2 is used, leaving a quotient of 2. Then, as previously explained, this completes the problem because the quotient (2) is a prime number.

Thus the prime factors of 16 are 2, 2, 2, and 2. You will note that the several divisors and the last quotient make up the complete list of factors.

2. Find the prime factors of 1716.

Solution

$$\begin{array}{r|l} 2 & 1716 \\ \hline 2 & 858 \\ \hline 3 & 429 \\ \hline 11 & 143 \\ \hline & 13 \end{array}$$

Prime factors are 2, 2, 3, 11, and 13.

The first step is to divide 1716 by the smallest prime number, above 1, which will divide into 1716 an exact number of times. First try 2. The 2 divides into 1716 exactly 858 times, so it is one of the factors.

Next find the smallest prime factor that will exactly divide into the quotient 858. Here again 2 is satisfactory and leaves a quotient of 429.

The next step is to find the smallest prime factor that will divide into 429 an exact number of times. The 2 cannot be used again because the quotient ends in 9 and thus is not exactly divisible by 2. Try 3. We find that 3 divides into 429 exactly 143 times, so it is satisfactory. The 3 will not divide an exact number of times into 143, so we try the next higher prime number, which is 5. This will not divide an exact number of times into 143, so we try the next higher prime factor, which is 7. This does not divide into 143 an exact number of times, so we try the next higher prime factor, which is 11. The 11 divides into 143 an exact number of times and leaves a quotient of 13. The 13 is a prime number, so the solution is complete.

Note: You will know when the last quotient becomes a prime number because if you cannot exactly divide it by any other number it is a prime number.

3. Find the prime factors of 2500.

Solution

$$
\begin{array}{r|r}
2 & 2500 \\ \hline
2 & 1250 \\ \hline
5 & 625 \\ \hline
5 & 125 \\ \hline
5 & 25 \\ \hline
 & 5
\end{array}
$$

Prime factors are 2, 2, 5, 5, 5, and 5.

4. Find the prime factors of 25,740.

Solution

$$
\begin{array}{r|r}
2 & 25740 \\ \hline
2 & 12870 \\ \hline
3 & 6435 \\ \hline
3 & 2145 \\ \hline
5 & 715 \\ \hline
11 & 143 \\ \hline
 & 13
\end{array}
$$

Prime factors are 2, 2, 3, 3, 5, 11, and 13.

Proving Factors. We can easily check our factors to see if they are correct by the simple process of multiplying all the factors. Their product should equal the number which was factored.

Take the preceding Example 1 for instance. The factors shown are 2, 2, 2, and 2. To check this example find the product of these factors.

$$2 \times 2 \times 2 \times 2 = 16$$

Thus the factors are correct.

To find the product of several factors, the process is as follows:

$$2 \times 2 \times 2 \times 2$$

$$
\begin{array}{r}
2 \\
2 \\ \hline
4 \\
2 \\ \hline
8 \\
2 \\ \hline
16 \text{ (product)}
\end{array}
$$

PRACTICE PROBLEMS

After you have solved the following problems, compare your answers with the correct answers shown on page 18.

1. Find the prime factors of 78.
2. Find the prime factors of 2772.
3. Find the prime factors of 311.
4. Find the prime factors of 4235.
5. Find the prime factors of 2310.
6. Find the prime factors of 1140.
7. Find the prime factors of 6578.
8. Find the prime factors of 189.
9. Find the prime factors of 1932.

Ordinary Factors. If prime factors are not required, the process of factoring can be made much easier by using ordinary factors. You learned in Lesson 1 that an ordinary factor could be *any* number. This means that in *ordinary* factoring we can use factors regardless of whether or not they are prime factors. In the prime factoring process, only one set of factors can be correct. In ordinary factoring, several sets may be correct. The reason why we can have several sets of correct ordinary factors is that we can start the factoring process using large, medium sized, or small factors. To find the ordinary factors of a number, we select any divisor which will divide into the number an exact number of times, and find the quotient. Then find a divisor which will divide into the quotient an exact number of times. Continue this process until a quotient is obtained which cannot be divided by any other number except itself and 1. The factors of the number include the several divisors and the last quotient.

ILLUSTRATIVE EXAMPLES

1. Factor the number 1716. (In problems where ordinary factors are required we say, "Factor the number" instead of "Find prime factors of.")

Solution

6)1716	4)1716	2)1716	13)1716
2) 286	3) 429	6) 858	6) 132
	13) 143	13) 143	2) 22
	11	11	11

In the four solutions above you can see that where ordinary factors are concerned we may have several sets of factors. Compare

this to Example 2 where the prime factor process is illustrated. You can see that the ordinary factoring process is shorter, and you will find it easier, too, because you are not held within the bounds of prime factors.

The process for finding ordinary factors is similar to that explained for finding prime factors except that, as shown above, we can use *any* number just so it divides an exact number of times into the number and quotients. The difference between prime factoring and ordinary factoring is simply that prime factoring requires that prime numbers be used for factors, whereas in ordinary factoring we may use *any* number.

2. Factor the number 2050.

Solution

10)2050	5)2050	2)2050	5)2050
5) 205	5) 410	5)1025	2) 410
41	2) 82	5) 205	5) 205
	41	41	41

In ordinary factoring it might happen that two or three people working the same problem would each use a different set of factors and each be correct. There is no standard rule to go by in finding ordinary factors except that in higher mathematics we may sometimes find one set of factors better than another for a particular purpose. However, for the present all you need be concerned about is learning the general method of factoring.

3. Factor the number 6040.

Solution

20)6040	5)6040	10)6040	2)6040
2) 302	4)1208	2) 604	10)3020
151	2) 302	2) 302	2) 302
	151	151	151

To prove any set of factors, find the product of the factors. The product should be the same as the number which was factored.

Common Factors. In your study of Lesson 1 you learned that if any integer divides an exact number of times into two or more numbers it is a common factor of those numbers.

Between common factors and prime or ordinary factors there is a distinct difference which you should understand at this point.

Prime or ordinary factors are for *one* number, whereas common factors are for *two or more* numbers.

To find the common factors for two or more numbers, we start out with 2 and then try 3, 4, 5, 6, 7, 8, etc., in turn until we have determined all the integers that will divide into the two or more numbers an exact number of times. There is a limit to the number of integers we try. Take the numbers 24 and 60. In this case 12 is the highest integer we would try because we cannot have a common factor that is more than one-half of the smaller of the two numbers.

ILLUSTRATIVE EXAMPLES

1. Find the common factors of 28 and 84.

Solution. First try 2. The 2 divides into *both* 28 and 84 an exact number of times, so it is one of the common factors. Next try 3. The 3 will not divide exactly into *both* 28 and 84 so it is not one of the factors. Next try 4. The 4 will divide an exact number of times into 28 and 84 so it is one of the common factors. Next try 5. It is not a factor because it will not divide into *both* 28 and 84 an exact number of times. Try 6. It is not a factor. Try 7. It exactly divides *both* 28 and 84 and is one of the common factors. In like manner try 8, 9, 10, 11, 12, 13, and 14. Of these integers only 14 divides both 28 and 84 an exact number of times. Thus 14 is one of the common factors. Fourteen is the last integer we need try because it is half of the smallest of the two numbers 28 and 84.

Thus the common factors of 28 and 84 are 2, 4, 7, and 14.

2. Find the common factors of 36, 48, and 72.

Solution. First try 2. Two will divide an exact number of times into all three numbers so it is a common factor. Next try 3. It will divide into all three numbers so it is a common factor. In like manner 4 is a common factor. Try 5. It will not divide an exact number of times into all three numbers so it is not a common factor. Try 6. It is a common factor. The integers 7, 8, 9, 10, and 11 are not common factors. Try 12. It is a common factor, but 13, 14, 15, 16, 17, and 18 are not. Here 18 is half of 36 (the smallest number) so it is the last integer we need try.

You will not have any trouble with common factors if you try one integer after another until you come to an integer which is half of the smallest of the numbers.

PRACTICE PROBLEMS

After you have solved the following problems, compare your answers with the correct answers on page 18.

1. Find the common factors of 16, 40, and 96.
2. Find the common factors of 40, 80, and 100.
3. Find the common factors of 31, 43, and 57.
4. Find the common factors of 36, 18, and 54.

Lesson 3

For Step 1, recall what you have learned in Lessons 1 and 2. Be sure you understand all three types of factoring. For Step 2, learn how to do cancellation. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

CANCELLATION

In Section 1 you learned that an expression, such as $8 \div 4$, indicates division and that 8 is to be divided by 4. Thus the 8 is the dividend and the 4 is the divisor. Now a new way of indicating division must be learned. If we write the dividend above and the divisor below a short horizontal line, we have $\frac{8}{4}$. This $8 \div 4$ and $\frac{8}{4}$ mean exactly the same thing. Whenever you see an expression like $\frac{8}{4}$, it means that the number above the line (dividend) is to be divided by the number below the line (divisor).

Now suppose we have the expression $96 \div 48$. We can also write this $\frac{96}{48}$. We could easily divide 96 by 48 using the methods learned in Section 1. There is another way of doing the same thing, using the principles of cancellation. First find the prime factors of both 96 and 48.

2)96	2)48
2)48	2)24
2)24	2)12
2)12	2) 6
2) 6	3
3	

Then write the factors of 96 as the dividend (above a horizontal line) and the factors of 48 as the divisor (below a horizontal line). A times sign ($\times$) is put between the factors because the product of the factors equals the numbers. (See *Proving Factors* in Lesson 2.) Thus we have—

$$\frac{96}{48} = \frac{2\times2\times2\times2\times2\times3 \text{ (dividend)}}{2\times2\times2\times2\times3 \text{ (divisor)}}$$

For convenience write the factors of the dividend (above line) and the divisor (below line) so that they are one directly above the other. If the dividend has more factors than the divisor each factor is indicated in its proper place above the line, as shown, and a blank left (below the line) where a divisor is omitted.

Cancellation is the process of dividing one factor in the dividend and one factor in the divisor by the same number. Where such a division is done, the number must divide into both factors an exact number of times. This process can be repeated for other factors as often as possible.

$$\overset{(a)}{\frac{2\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{3}}}} = \overset{(b)}{\frac{2\times1\times1\times1\times1\times1}{1\times1\times1\times1\times1}} = \overset{(c)}{\frac{2}{1}} \overset{(d)}{=2}$$

At (*a*) you can see there are several 2's in both the dividend and the divisor. We can select any 2 in the dividend and any 2 in the divisor and divide both by 2 an exact number of times. We can do this four times because there are four 2's in the divisor and five 2's in the dividend. In each of these cases 2 divides into 2 once. So we cross out the 2's and place a 1 above and below as shown. There is a 3 in both the dividend and divisor. Both of these 3's can be divided exactly by 3 with a quotient of 1. So the 3's are crossed out and a figure 1 is substituted for each. One figure, 2, is left in the dividend. However, there are no figures, above 1, in the divisor, so the dividing is complete.

At (*b*) we have written the results of the division at (*a*). We next find the product of the figures left in the dividend and divisor and write these products as shown at (*c*). The $\frac{2}{1}$ means $2\div1$ so the answer, at (*d*), is 2.

Sometimes in problems we have a group of factors forming a dividend and another group forming the divisor, such as

$$\frac{18\times5 \text{ (dividend)}}{10\times3 \text{ (divisor)}}$$

We could find the products of 18×5 and 10×3 and divide the product of the dividend by the product of the divisor, but an easier method of solution is by cancellation.

(*a*) (*b*) (*c*) (*d*)

$$\frac{\overset{\overset{3}{\cancel{6}}}{\cancel{18}}\times\overset{1}{\cancel{5}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{10}}\times\underset{1}{\cancel{3}}}=\frac{3\times1}{1\times1}=\frac{3}{1}=3\ \text{(Ans.)}$$

At (*a*) we can cancel by dividing dividend and divisor numbers (factors) by the same number, as already explained. The 10 in the divisor and the 5 in the dividend can both be divided an exact number of times by 5. So we cross out 10 and write 2, and cross out the 5 and write 1. Both 18 and 3 can be exactly divided by 3. So we cross out the 18 and write 6, because 3 goes into 18 exactly 6 times. Then we cross out the 3 and put 1 in its place. From previous divisions we had a 2 under the 10 and a 6 over the 18. Both the 2 and 6 can be exactly divided by 2. So we cross out the 6 and write 3, and cross out the 2 and write 1.

What is left of the dividend and divisor is shown at (*b*). At (*c*) are the products of dividend and divisor. At (*d*) is the result of $\frac{3}{1}=3\div1=3$.

Notice. Be sure you can work all examples without looking at the book.

ILLUSTRATIVE EXAMPLES

1. Cancel $\frac{6\times8\times12\times18\times24}{2\times3\times\ 2\times\ 4\times\ 6}=?$

Solution

Instruction — *Operation*

Step 1

The 18 in the dividend and the 6 in the divisor can both be divided an exact number of times by 6. The $6\div6=1$ and the $18\div6=3$. Thus 18 and 6 are crossed out and 3 and 1 are written in their places.

$$\frac{6\times8\times12\times\cancel{18}\times24}{2\times3\times\ 2\times\ 4\times\ \underset{1}{\cancel{6}}}$$

Step 2

The 12 above the line and the 2 below the line can both be divided an exact number of times by 2. This leaves 6 above the line and 1 below the line.

$$\frac{6\times8\times\overset{6}{\cancel{12}}\times\overset{3}{\cancel{18}}\times24}{2\times3\times\ \underset{1}{\cancel{2}}\times\ 4\times\ \underset{1}{\cancel{6}}}$$

Instruction	*Operation*
Step 3	**Step 3**
The 24 above the line and the 4 below the line can both be divided an exact number of times by 4. This leaves 6 and 1.	$\dfrac{6\times8\times\overset{6}{\cancel{12}}\times\overset{3}{\cancel{18}}\times\overset{6}{\cancel{24}}}{2\times3\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{4}}\times\underset{1}{\cancel{6}}}$
Step 4	**Step 4**
A 6 above the line and the 3 below the line can both be divided by 3 an exact number of times. This leaves 2 in place of 6 and 1 in place of 3.	$\dfrac{\overset{2}{\cancel{6}}\times8\times\overset{6}{\cancel{12}}\times\overset{3}{\cancel{18}}\times\overset{6}{\cancel{24}}}{2\times\underset{1}{\cancel{3}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{4}}\times\underset{1}{\cancel{6}}}$
Step 5	**Step 5**
The 8 above the line and the 2 below the line can both be divided an exact number of times by 2. This leaves 4 in place of 8 and 1 in place of 2.	$\dfrac{\overset{2}{\cancel{6}}\times\overset{4}{\cancel{8}}\times\overset{6}{\cancel{12}}\times\overset{3}{\cancel{18}}\times\overset{6}{\cancel{24}}}{\underset{1}{\cancel{2}}\times\underset{1}{\cancel{3}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{4}}\times\underset{1}{\cancel{6}}}$
Step 6	**Step 6**
In Step 5 the last factor in the divisor was cancelled, so no more division is possible. In this step is shown what is left after all cancellation is done. Study what is shown here and you will see that the factors remaining in Step 5 are shown in this step at (*a*). At (*b*) are shown the products of dividend and divisor factors. The $\frac{864}{1}$ is the same as $864\div1$ so the answer (*c*) is 864.	$\underset{(a)}{\dfrac{2\times4\times6\times3\times6}{1\times1\times1\times1\times1}}=\underset{(b)}{\dfrac{864}{1}}=\underset{(c)}{864}$

Note. By this time you have seen that *any* factor in the dividend and *any* factor in the divisor can be cancelled. The factors do not have to be one above the other or separately represented.

2. Find the quotient of $\frac{24\times720}{4\times12\times\ 60}$.

Solution

Instruction	*Operation*

Step 1

Finding the Quotient and **Cancel** mean the same thing. So we do the same in this example as in Example 1. The 24 above the line and the 12 below the line can both be divided an exact number of times by 12. Cross out 24 and write 2. Cross out 12 and write 1.

Step 1

$$\frac{\overset{2}{\cancel{24}}\times 720}{4\times\underset{1}{\cancel{12}}\times\ 60}$$

Step 2

The figures 720 and 60 can both be exactly divided by 10. This leaves 72 and 6. Then both 72 and 6 can be exactly divided by 6. This leaves 12 and 1.

Step 2

$$\frac{\overset{2}{\cancel{24}}\times \overset{\overset{12}{\cancel{72}}}{\cancel{720}}}{4\times\underset{1}{\cancel{12}}\times \underset{\underset{1}{\cancel{6}}}{\cancel{60}}}$$

Step 3

The 12 above the line and 4 below the line can both be exactly divided by 4. This leaves 3 and 1. This completes the cancelling.

Step 3

$$\frac{\overset{2}{\cancel{24}}\times \overset{\overset{\overset{3}{\cancel{12}}}{\cancel{72}}}{\cancel{720}}}{\underset{1}{\cancel{4}}\times\underset{1}{\cancel{12}}\times \underset{\cancel{6}}{\cancel{60}}}$$

Step 4

The factors left, after cancelling, are 2 and 3 above the line and 1, 1, and 1 below the line. This is shown at (*a*). The product of factors is $\frac{6}{1}$ as shown at (*b*). The $\frac{6}{1}=6\div1=6$ as shown at (*c*).

Step 4

$$\overset{(a)}{\frac{2\times3}{1\times1\times1}}=\overset{(b)}{\frac{6}{1}}=\overset{(c)}{6}\ \text{(Ans.)}$$

Note: There are many different cancellations which could have been used in both Examples 1 and 2. All would be correct if the answers are all the same. You can cancel Examples 1 and 2 in many different ways and still get the same answer. Try it.

3. Cancel $\frac{16\times 2\times 15\times 4}{32\times 6\times 8\times 22}$.

Solution. Detailed explanation is not given here but all the operations are shown.

Step 1
$$\frac{\overset{1}{\cancel{16}}\times 2\times 15\times 4}{\underset{2}{\cancel{32}}\times 6\times 8\times 22}$$

Step 2
$$\frac{\overset{1}{\cancel{16}}\times 2\times \overset{5}{\cancel{15}}\times 4}{\underset{2}{\cancel{32}}\times \underset{2}{\cancel{6}}\times 8\times 22}$$

Step 3
$$\frac{\overset{1}{\cancel{16}}\times \overset{1}{\cancel{2}}\times \overset{5}{\cancel{15}}\times 4}{\underset{2}{\cancel{32}}\times \underset{\underset{1}{\cancel{2}}}{\cancel{6}}\times 8\times 22}$$

Step 4
$$\frac{\overset{1}{\cancel{16}}\times \overset{1}{\cancel{2}}\times \overset{5}{\cancel{15}}\times \overset{1}{\cancel{4}}}{\underset{2}{\cancel{32}}\times \underset{\underset{1}{\cancel{2}}}{\cancel{6}}\times \underset{2}{\cancel{8}}\times 22}$$

Step 5
$$\frac{1\times 1\times 5\times 1}{2\times 1\times 2\times 22}=\frac{5}{88}\text{ (Ans.)}$$

Note: The answer $\frac{5}{88}$ cannot be carried any further because you have not studied the required principles. Therefore we will leave it as $\frac{5}{88}$. In like manner if any other examples have such answers, just leave them at that stage. In sections to follow you will learn how to handle such expressions.

4. Cancel $\frac{16\times 12\times 7\times 11}{24\times 22\times 7\times 18}$.

Solution. Here only a combined solution and answer are shown. See if you can solve the problem and get the same answer. Do not worry if your cancellation is different.

$$\frac{\overset{2}{\cancel{16}}\times \overset{\overset{1}{\cancel{2}}}{\cancel{12}}\times \overset{1}{\cancel{7}}\times \overset{1}{\cancel{11}}}{\underset{3}{\cancel{24}}\times \underset{\underset{1}{\cancel{2}}}{\cancel{22}}\times \underset{1}{\cancel{7}}\times \underset{3}{\cancel{18}}}=\frac{2}{9}\text{ (Ans.)}$$

Proving Cancellation. There is no method of proving cancellation except to be very careful not to make errors in division, or to work the same problem two or three times to make sure you have made no mistakes and that you get the same answer each time. Cancellation is not at all difficult if you make it a rule to be very careful and to at least work each problem twice for the sake of proving your answer.

PRACTICE PROBLEMS

After you have worked all of the following problems, compare your answers with the correct answers shown on page 18.

If you have any trouble getting the correct answers it is because you have made an error somewhere in your division. This means you should work out the incorrect problems several times or until you find your mistake.

Find the quotient of the following.

1. $\frac{6\times12\times20}{4\times8\times5}$

2. $\frac{16\times24\times18}{36\times4\times6}$

3. $\frac{12\times60\times36\times70}{28\times5\times48\times6}$

4. $\frac{16\times72\times48\times32}{8\times24\times16\times36}$

5. $\frac{72\times70\times96\times100\times150}{25\times16\times10\times36\times50}$

6. $\frac{64\times24\times85\times32\times64}{40\times32\times64\times8\times34}$

7. $\frac{48\times940}{16\times94\times80}$

8. $\frac{44\times75\times63\times56\times45}{16\times35\times50\times55\times18}$

9. $\frac{32\times24\times14\times22}{24\times44\times14\times18}$

10. $\frac{24\times20\times36\times150}{12\times6\times5\times30}$

11. $\frac{25\times16\times12}{10\times4\times6\times7}$

12. $\frac{120\times44\times6\times7}{72\times33\times14}$

13. $\frac{140\times39\times13\times7}{30\times7\times26\times21}$

14. $\frac{200\times36\times30\times21}{270\times40\times15\times14}$

15. $\frac{18\times6\times4\times42}{4\times9\times3\times7}$

ANSWERS TO PRACTICE PROBLEMS

Lesson 1, Page 4

1. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. **2.** 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41. **3.** 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60. **4.** 13, 17, 19, 23, 29, 31, 37, 41, 43. **5.** Common factors. **6.** They are prime to each other. **7.** Factors.

Lesson 2, Page 8

1. 2, 3, 13. **2.** 2, 2, 3, 3, 7, 11. **3.** Prime number—has no factors. **4.** 5, 7, 11, 11. **5.** 2, 3, 5, 7, 11. **6.** 2, 2, 3, 5, 19. **7.** 2, 11, 13, 23. **8.** 3, 3, 3, 7. **9.** 2, 2, 3, 7, 23.

Lesson 2, Page 11

1. 2, 4, 8. **2.** 2, 4, 5, 10, 20. **3.** None. **4.** 2, 3, 6, and 9.

Lesson 3, Page 17

1. 9. **2.** 8. **3.** 45. **4.** 16. **5.** 1,008. **6.** 12. **7.** $\frac{3}{8}$. **8.** $\frac{189}{10}$. **9.** $\frac{8}{9}$. **10.** 240. **11.** $\frac{20}{7}$. **12.** $\frac{20}{3}$. **13.** $\frac{13}{3}$. **14.** 2. **15.** 24.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or final examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on page 21.

If you miss more than two of the problems it means you should review Lessons 1, 2, and 3 carefully.

Do not try this trial examination until you have worked every practice problem in Lessons 1, 2, and 3.

Do not start the final examination until you have completed this trial examination.

1. Write the odd numbers between 40 and 70.
2. Write the even numbers from 30 to 50.
3. Write the prime numbers between 15 and 40.
4. Find the prime factors of 2720.
5. Find the prime factors of 6450.
6. Find the prime factors of 3480.
7. Cancel $\frac{8\times 10\times 24}{5\times 4\times 16}=?$
8. Cancel $\frac{90\times 66\times 8}{4\times 11\times 30}$
9. Find the quotient of $\frac{21\times 11\times 26}{14\times 13}$
10. Find the quotient of $\frac{66\times 9\times 18\times 5}{22\times 6\times 40}$
11. What are the common factors of 40 and 90?
12. What are the common factors of 36, 72, 96?
13. What are the three prime factors of 105?
14. Factor into prime factors and then divide 2912 by 1456 using cancellation.
15. Factor into prime factors and then divide by cancellation, 79856 ÷ 5704.

FINAL EXAMINATION

1. Write the odd numbers between 1 and 19.

2. Write the even numbers between 19 and 30.

3. Write the prime numbers between 18 and 36.

4. Find the prime factors of 2310.

5. Find the prime factors of 2508.

6. Cancel $\frac{2\times4\times7}{2\times7\times2}=?$

7. Cancel $\frac{3\times8\times\ \ 9\times4}{2\times27\times8}=?$

8. Cancel $\frac{12\times10\times18\times75}{6\times\ \ 3\times\ \ 5\times15}=?$

9. Cancel $\frac{8\times\ \ 6\times11\times13}{12\times22\times13\times\ \ 9}=?$

10. Find the quotient of $\frac{40\times48\times54\times60}{30\times24\times72\times\ \ 3}$

11. Find the quotient of $\frac{27\times33\times75\times56}{54\times11\times15\times\ \ 8}$

12. What are the three prime factors of 539?

13. What are the common factors of 24, 60, and 96?

14. Factor into prime factors and then divide by cancellation, $73920 \div 10560$.

15. What prime factor is also an even number?

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69.

If you remember the definition of an odd number from Lesson 1 you should not have had any trouble because odd numbers are those which cannot be divided an exact number of times by 2.

2. 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50.

Even numbers are those which can be divided by 2 an exact number of times.

3. 17, 19, 23, 29, 31, 37.

Prime numbers are those which can only be divided an exact number of times by themselves and 1.

4.

```
2)2720
2)1360
2) 680
2) 340
2) 170
5)  85
    17
```

Remember the explanation given in Lesson 2. Start dividing, using smallest prime number that will divide an exact number of times into 2,720. Then divide the quotient by the smallest prime number that will divide it an exact number of times, etc. The quotient 17 is a prime number so it becomes the last quotient.

5.

```
2)6450
3)3225
5)1075
5) 215
    43
```

6.

```
2)3480
2)1740
2) 870
3) 435
5) 145
    29
```

7.

$$\frac{\overset{2}{\cancel{8}}\times\overset{\overset{1}{\cancel{2}}}{\cancel{10}}\times\overset{3}{\cancel{24}}}{\underset{1}{\cancel{5}}\times\underset{1}{\cancel{4}}\times\underset{\underset{1}{\cancel{2}}}{\cancel{16}}}=\frac{2\times1\times3}{1\times1\times1}=\frac{6}{1}=6 \text{ (Ans.)}$$

There are many possible ways of cancelling. The one shown above and explained below is typical. There can be only one correct answer. The cancellation was done as follows:

Divide 24 (above the line) and 16 (below the line) by 8, leaving 3 and 2.
Divide 10 (above the line) and 5 (below the line) by 5, leaving 2 and 1.
Divide 8 (above the line) and 4 (below the line) by 4, leaving 2 and 1.
Divide 2 (above the line) and 2 (below the line) by 2, leaving 1 and 1.

After cancellation is done, we have $2\times1\times3$ above the line and $1\times1\times1$ below the line. This is written

$$\frac{2\times1\times3}{1\times1\times1}$$

Multiplying $2\times1\times3$ (above the line) gives 6.
Multiplying $1\times1\times1$ (below the line) gives 1.
Then we have $\frac{6}{1}$. We know this means $6\div1$ and that $6\div1=6$. Ans.

8.
$$\frac{\overset{3}{\cancel{90}}\times\overset{6}{\cancel{66}}\times\overset{2}{\cancel{8}}}{\underset{1}{\cancel{4}}\times\underset{1}{\cancel{11}}\times\underset{1}{\cancel{30}}}=\frac{3\times6\times2}{1\times1\times1}=\frac{36}{1}=36$$

Remember that there are several ways of cancelling but there is only one right answer.

9.
$$\frac{\overset{3}{\cancel{21}}\times11\times\overset{\overset{1}{\cancel{2}}}{\cancel{26}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{14}}\times\underset{1}{\cancel{13}}}=\frac{3\times11\times1}{1\times1}=\frac{33}{1}=33$$

10.
$$\frac{\overset{3}{\cancel{66}}\times9\times\overset{3}{\cancel{18}}\times\overset{1}{\cancel{5}}}{\underset{1}{\cancel{22}}\times\underset{1}{\cancel{6}}\times\underset{8}{\cancel{40}}}=\frac{3\times9\times3\times1}{1\times1\times8}=\frac{81}{8}$$

11. Answer is 2, 5, and 10.
Remember to start with 2 and try each succeeding higher number up to and including a number equal to half of 40.

12. Answer is 2, 3, 4, 6, and 12.
Try all numbers from 2 to 18 and see which of them divide into 36, 72, and 96 an exact number of times. Those that do are common factors.

13.

3)105
5) 35
7

Answer = 3, 5, and 7.

14. Before we can use cancellation we must factor the numbers 2912 and 1456. We will find the prime factors.

2)2912
2)1456
2) 728
2) 364
2) 182
7) 91
13

2)1456
2) 728
2) 364
2) 182
7) 91
13

$$\frac{2\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{7}}\times\overset{1}{\cancel{13}}}{\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{7}}\times\underset{1}{\cancel{13}}}=\frac{2\times1\times1\times1\times1\times1\times1}{1\times1\times1\times1\times1\times1}=\frac{2}{1}=2$$

The answer is 2.

15.

```
  2)79856        2)5704
  2)39928        2)2852
  2)19964        2)1426
  2) 9982       23) 713
  7) 4991            31
 23)  713
       31
```

$$\frac{2\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times7\times\overset{1}{\cancel{23}}\times\overset{1}{\cancel{31}}}{\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{23}}\times\underset{1}{\cancel{31}}}=\frac{2\times1\times1\times1\times7\times1\times1}{1\times1\times1\times1\times1}=\frac{14}{1}=14$$

PRACTICAL MATHEMATICS

Section 3

FRACTIONS—Part I

In Section 2 you learned what a **whole number** is. You learned that a whole number and an **integer** are the same thing. Thus we know that any number such as 3, 5, 18, 25, 87, etc., is a whole number. Sometimes such numbers may be called by their other name (integer), but in this section we will use the name "whole number." If you are in any doubt as to the meaning of "whole number" be sure to review Section 2 before going any further in this section.

The new principles which you will study in this section go a step beyond whole numbers and explain "parts" of whole numbers.

In explaining the meaning of the word "part" or "parts" several illustrations can be given. First suppose you have an apple. The apple is whole or undivided. Thus this apple compares to a whole number. If you cut the apple into 4 equal pieces it is no longer a whole apple. Thus each of the 4 pieces becomes a "part" of the apple and all 4 pieces are "parts." Next think of an ordinary 12-inch rule or ruler. Such a rule is 1 foot long. Or, the entire rule is a whole foot. Rulers are divided into 12 equal divisions called inches. Each of these inches is a "part" of a foot and all 12 inches are "parts" of a foot. Now look at this book you are reading. As it is bound it is a whole book. However, the whole book contains several pages. Each page is a "part" of the whole book.

These "parts" which we have been thinking about can be given names depending on the number of the parts. Think of the apple which we cut into 4 equal parts. Each of these 4 parts is **one** "part" of the apple. Then because there are 4 "parts" each of the "parts" can be called **one-fourth** of the apple. Next think of the ruler which is divided into 12 "parts." Each of these "parts" is **one-twelfth** of a foot. Also, if our book contains 30 pages each page is one part or **one-thirtieth** of the book.

In like manner if we think of anything being divided into a certain number of equal parts we call each "part" **one=_______.** (*Write total number of parts in blank space.*)

ILLUSTRATIVE EXAMPLES

1. Suppose we divided a steel rod into 19 equal parts. What would we call one of these parts?

Answer. Each of these 19 equal parts was one part of the steel rod. Thus if there are 19 equal parts one of these parts is called **one=nineteenth.**

2. A pie is cut so as to make 6 equal parts. What would we call one of these parts?

Answer. Each piece is one of six equal parts, or **one=sixth.**

Fractions. Up to this point you have learned that anything can be divided into equal parts. You have also learned how to name one of these equal parts. In mathematics we have a special name for these parts. They are called **fractions.** Thus, in Example 1, the answer of one-nineteenth is called a fraction.

The word "fraction" means one part of an integer or whole number. This word also means "small." We can easily understand how these two meanings apply to what we have already learned. Think of the apple which we assumed was cut into 4 parts. Each of these 4 parts is a **fraction** because it is a part of a whole apple and because it is smaller than the whole. The apple itself is a **whole** just like 12 or 86 is a whole number. Each part is smaller than the whole apple. Therefore, we can see why these parts are **fractions.** This word will be used from now on.

Whole numbers can be divided into equal parts the same as the apple, the ruler, or the book, which we have already considered. Take the number 1 for example. If we divide this number into 4 equal parts, each part is **one=fourth** just as in the case of the whole apple cut into 4 equal parts. This **one=fourth** is a fraction. When we divide the number 1 into any number of equal parts we cannot actually see those parts as we could see the parts of an apple. So when we think of the number 1 being divided into parts, we have to see these parts in our minds, that is, imagine them. We can, however, substitute some simple illustrations in place of the number 1 and thus help ourselves to see how 1 can be divided into 2 or more parts.

See Fig. 1. Here we have drawn a square. Think of this as representing 1 square or a whole number. In whatever way you think of Fig. 1 remember that it stands for 1.

See Fig. 2. Here we have shown the same block as in Fig. 1 except that it has been divided into 2 equal parts. Here each part is half of the block so each part is called **one=half.**

See Fig. 3. Here we took the same block shown in Fig. 1 and divided it into 4 equal parts. Each part is therefore **one=fourth.**

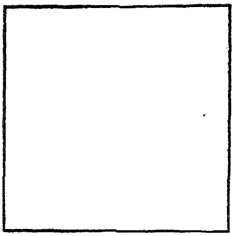


Fig. 1

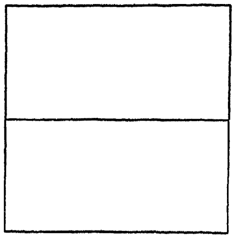


Fig. 2

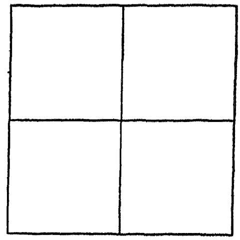


Fig. 3

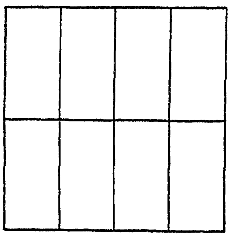


Fig. 4

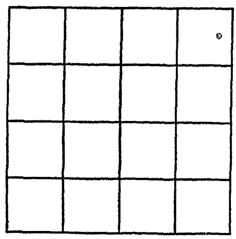


Fig. 5

See Fig. 4. Here we took the same block as in Fig. 1 and divided it into 8 equal parts. Each part is **one=eighth.**

See Fig. 5. Here we took the same block as in Fig. 1 and divided it into 16 equal parts. Each part is **one=sixteenth.**

We could go on dividing the block in Fig. 1 into more and more equal parts. However, enough have been shown to illustrate that the number 1 can be divided into any number of equal parts. In every case such as Figs. 2, 3, 4, and 5 each part is a **fraction** of the whole block or of the number 1.

Having learned what "parts" and "fractions" are, there is an important principle which you must keep in mind.

Rule 1. *When you think of or work with fractions, you must remember that these fractions are parts of 1.*

The number 1 is the smallest whole number. Thus anything smaller than one is a fraction and, as shown in Figs. 1 through 5, the number 1 can be divided into any number of "fractional parts."

We have been writing fractions in terms of **one-fourth, one-twelfth, one-eighth, one-sixteenth,** etc. There is a much shorter and simpler way to write these same fractions so that nothing but numbers are used.

For example

one-half is written as $\frac{1}{2}$
one-third is written as $\frac{1}{3}$
one-fourth is written as $\frac{1}{4}$
one-fifth is written as $\frac{1}{5}$
one-sixth is written as $\frac{1}{6}$
one-seventh is written as $\frac{1}{7}$
one-eighth is written as $\frac{1}{8}$
one-ninth is written as $\frac{1}{9}$
one-tenth is written as $\frac{1}{10}$

These examples will serve to illustrate how fractions should be written.

We will examine one of these fractions written in this way and see what it means. Take the fraction $\frac{1}{4}$ for example.

$$\frac{1}{4} \quad \frac{\text{(numerator)}}{\text{(denominator)}}$$

Here we must learn some new names. Each part of a fraction has a name, as shown above.

The portion above the horizontal line is called the **numerator.**

The portion below the horizontal line is called the **denominator.**

Refer to Fig. 3. Here we divided the square into 4 equal parts. We learned that one of these parts is called *one-fourth.* Also we have learned that in mathematics we write *one-fourth* as $\frac{1}{4}$.

Now we have learned that the denominator shows into how many parts a unit (whole) has been divided. The numerator shows how many of these parts we have in mind. See Fig. 6. Here we see another square divided into 4 equal parts, as in Fig. 3. One of the 4 parts is shaded; let the fraction $\frac{1}{4}$ represent this portion. Thus, the denominator of the fraction $\frac{1}{4}$ indicates that we divided the block

in Fig. 6 into 4 parts and the numerator indicates that we are thinking of 1 of those parts.

Refer back to Fig. 5. Here we divided the block into 16 equal parts. If we are thinking of one of these 16 parts the fraction which represents our thought is $\frac{1}{16}$.

Now suppose we were thinking of 3 of the parts in Fig. 5. Then our fraction would be $\frac{3}{16}$. If we think of 12 of the parts in Fig. 5 the fraction is $\frac{12}{16}$.

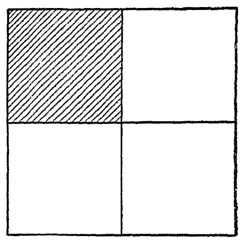


Fig. 6

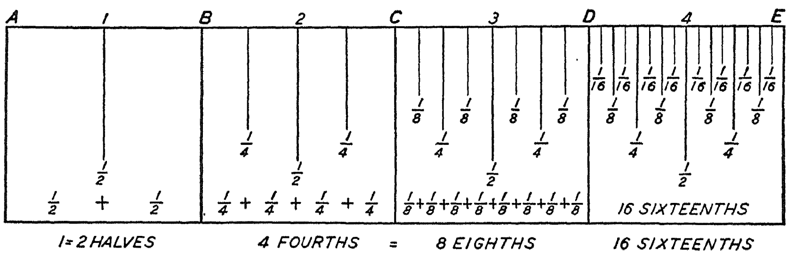


Fig. 7

As another example we can use a common ruler. Fig. 7 shows part of a ruler. From A to E the distance is exactly 4 inches. The distance AB indicates 1 inch. The distance BC indicates the second inch, etc. The numbers *1*, *2*, *3*, and *4* simply number the inches. Now let us study this ruler from the standpoint of fractions.

The first inch is divided into two equal parts. Each of these two parts is one-half inch. Thus each part would be written as $\frac{1}{2}$. The denominator 2 indicates into how many parts the inch was divided, while the numerator shows that we are thinking of only one of those parts.

The second inch is divided into four equal parts, so each part is one-fourth inch and is written $\frac{1}{4}$. The denominator 4 indicates

the number of parts the inch is divided into, while the numerator 1 shows how many parts we are thinking of. You will see by the sketch that there are two fourths for each half inch.

In the third inch there are eight equal divisions, which are called eighths, each of which is written $\frac{1}{8}$. You can see that there are now four eighths where before there were only two fourths, or one half.

In the fourth inch in the sketch there are sixteen equal divisions, called sixteenths, each of which is written $\frac{1}{16}$. As in the previous cases the denominator 16 indicates how many parts are in the inch.

There is something more we can learn from Fig. 7.

The first inch is divided into two equal parts or halves. We learned that each such part is written $\frac{1}{2}$. You can also see that two of these $\frac{1}{2}$ fractions equals 1 inch. In other words $\frac{1}{2}+\frac{1}{2}=1$.

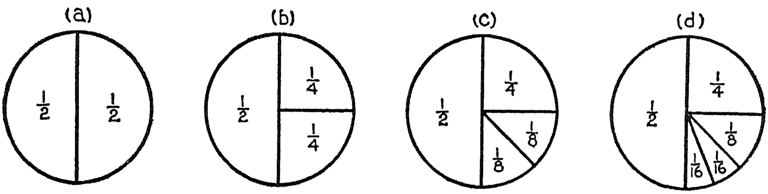


Fig. 8

In the second inch there are four one-quarters ($\frac{1}{4}$). Thus we can see that $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=1$. Studying the second inch we can also see that $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

In the third inch we see that the eight $\frac{1}{8}$ spaces when added equal 1. We see, too, that $\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$.

In the fourth inch we can see that the sixteen $\frac{1}{16}$ spaces when added equal 1. We see that $\frac{1}{16}+\frac{1}{16}=\frac{1}{8}$.

Study Fig. 7 carefully until you understand the preceding 4 paragraphs. We used the number 1. This meant 1 inch but while we were showing that $\frac{1}{2}$ inch added to $\frac{1}{2}$ inch equals 1 inch, we were also showing that $\frac{1}{2}+\frac{1}{2}$ equals 1 without thinking in terms of inches. (You should understand from this that **Rule 1** has also been explained).

As a further example study Fig. 8. Parts of different shape are illustrated by cutting a round cake into various pieces. In (*a*) the cake is cut into two equal portions and, as in Fig. 2, the result is two equal parts, called halves.

In (*b*) one of the halves of the cake is cut into two equal parts and these parts are called fourths, as in Fig. 3. Thus you can see one half equals two fourths and is written $\frac{1}{2}=\frac{2}{4}$.

In (*c*) one of the fourths is divided into two equal parts and these parts are called eighths, as in Fig. 4. Here the shapes of the sections are different, but the number of parts are the same. Thus one fourth equals two eighths and is written $\frac{1}{4}=\frac{2}{8}$.

In (*d*) one of the eighths is divided into two equal parts and these parts are called sixteenths, the same as in Fig. 5. Here again the parts differ in shape, but the number of parts is the same. Thus one eighth is equal to two sixteenths and is written $\frac{1}{8}=\frac{2}{16}$. You can also see by (*d*) that one-fourth equals all of the smaller parts in the other one-fourth of the cake. Therefore $\frac{1}{4}=\frac{1}{8}+\frac{1}{16}+\frac{1}{16}$ as in Fig. 7 which illustrates the divisions of the ruler.

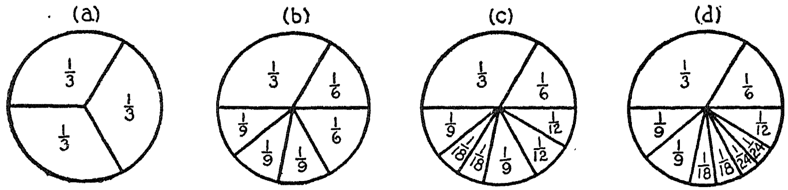


Fig. 9

In (*d*) which piece of cake would you choose if you were hungry? You would not choose the $\frac{1}{16}$ even if the denominator has a large number, for the large denominator with a smaller numerator indicates that the piece is small. You would choose the one-half, for it equals all the other parts put together, or $\frac{1}{2}=\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}$, as shown in (*d*).

Other fractions can be obtained by dividing the cake in another way. Study carefully Fig. 9. In (*a*) the cake is divided into three equal parts, so each part is one third and is written $\frac{1}{3}$. As in the other examples the denominator indicates into how many parts the cake is divided.

In (*b*) one of the thirds is divided into two equal parts, and each part becomes one sixth ($\frac{1}{6}$); this shows that $\frac{1}{3}=\frac{2}{6}$. Another third is divided into three equal parts and each part becomes one ninth; this shows that $\frac{1}{3}=\frac{3}{9}$. Thus the denominator indicates into how many parts the cake would be divided if all the parts were of this size.

In (*c*) one of the sixths is divided into two equal parts and each part becomes one twelfth ($\frac{1}{12}$); this shows $\frac{1}{6}=\frac{2}{12}$. One of the ninths in (*c*) is also cut into two equal parts and each part becomes one eighteenth ($\frac{1}{18}$), so $\frac{1}{9}=\frac{2}{18}$.

In (*d*) one of the twelfths is divided into two equal parts and each part becomes one twenty-fourth ($\frac{1}{24}$); this shows that $\frac{1}{12}=\frac{2}{24}$.

By comparing the four sketches of Fig. 9 and noting the results obtained, you can see that $\frac{1}{3}=\frac{2}{6}$, for $\frac{1}{3}$ was divided into two one-sixths, and another $\frac{1}{3}$ was divided into three one-ninths.

Study Fig. 9 and you will see that the following examples are true.

$$(1)\ \frac{1}{3}=\frac{1}{6}+\frac{1}{12}+\frac{1}{12}$$

Explanation

Look at (*c*) in Fig. 9. Here we see that $\frac{1}{6}$ together with two of the $\frac{1}{12}$ parts is equal to the $\frac{1}{3}$ part. So you can see at once that $\frac{1}{3}$ is equal to $\frac{1}{6}+\frac{1}{12}+\frac{1}{12}$.

Remember that just because the denominators of the $\frac{1}{6}+\frac{1}{12}+\frac{1}{12}$ are all larger numbers than the denominator of $\frac{1}{3}$ it does not mean that they are greater in value than $\frac{1}{3}$. The number in the denominator simply shows into how many parts the cake, in this case, has been divided. You can see readily that $\frac{1}{3}$ is the same as $\frac{1}{6}+\frac{1}{12}+\frac{1}{12}$.

$$(2)\ \frac{1}{3}=\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\frac{1}{24}$$

Explanation

Look at (*d*) in Fig. 9. You can see that together the $\frac{1}{6}$ and $\frac{1}{12}$ and two of the $\frac{1}{24}$th's take up the same space as does $\frac{1}{3}$.

Study Fig. 9 until you understand the following examples:

(3) $\frac{1}{3}=\frac{1}{9}+\frac{1}{9}+\frac{1}{18}+\frac{1}{18}$
(4) $\frac{1}{6}=\frac{1}{12}+\frac{1}{12}$
(5) $\frac{1}{12}=\frac{1}{24}+\frac{1}{24}$
(6) $\frac{1}{3}=\frac{2}{6}$ or $\frac{3}{9}$ or $\frac{4}{12}$ or $\frac{6}{18}$ or $\frac{8}{24}$
(7) $\frac{2}{3}=\frac{4}{6}$ or $\frac{6}{9}$ or $\frac{8}{12}$ or $\frac{12}{18}$ or $\frac{16}{24}$
(8) $\frac{3}{3}=\frac{6}{6}$ or $\frac{9}{9}$ or $\frac{12}{12}$ or $\frac{18}{18}$ or $\frac{24}{24}$

REDUCTION OF FRACTIONS

In mathematics the word *reduction* means the same as *change* or *changing*. Thus **reduction of fractions** simply means to change them

from one form to another form. There are many such changes necessary in ordinary use of mathematics.

The word "Terms," as used in mathematics, is a name given to the numerator and denominator of a fraction. Thus if we speak of the *terms* of a fraction we mean the numerator and the denominator considered together.

Lesson 1

For Step 1, recall what you know about fractions. For Step 2, learn the fundamental principles of reduction of fractions. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Reducing (Changing) Fractions to Higher Terms. Reducing or changing fractions to higher terms simply means that both the numerator and denominator are multiplied by the same number. Suppose we take the fraction $\frac{1}{3}$ and reduce it to higher terms.

Rule 2. *To reduce a fraction to higher terms, multiply both the numerator and denominator by the same number.*

Following the rule, we select the number 2 and multiply both the 1 and the 3, of the $\frac{1}{3}$, by this 2. This gives us two sixths. This is expressed mathematically in the following manner:

$$\frac{1 \times 2}{3 \times 2} = \frac{2}{6}$$

This $\frac{2}{6}$ has exactly the same value as $\frac{1}{3}$. To prove this refer to Fig. 9 and note part (*b*). Here we see that two of the $\frac{1}{6}$ parts taken together make a part the same size as the $\frac{1}{3}$ part.

This shows that when we reduce a fraction to higher terms we do not change its value at all. When we reduced $\frac{1}{3}$ to $\frac{2}{6}$ it had exactly the same value. In reducing a fraction to higher terms we simply divide it into more parts. In other words there are a *higher* or greater number of parts, but no change in actual value. Thus if we had $\frac{1}{3}$ of a cake and reduced this $\frac{1}{3}$ to $\frac{2}{6}$ we would still have the same amount of cake.

Now reduce $\frac{1}{9}$ to higher terms. Multiply both numerator and denominator by 2.

$$\frac{1 \times 2}{9 \times 2} = \frac{2}{18}$$

To prove this refer to Fig. 9 (*c*). Here we see that two of the $\frac{1}{18}$ parts are exactly the same size as one of the $\frac{1}{9}$ parts.

You may wonder what rule to use in deciding the number to multiply by when following **Rule 2.** The answer is that you can use any convenient number. In higher mathematics you will learn when to use a definite number. For the present just remember **Rule 2.**

Reducing a Fraction to Higher Terms with a Given Denominator. Sometimes in mathematical work we want to reduce a fraction to higher terms so that the new denominator will be a given number. For example, suppose we wish to reduce $\frac{1}{2}$ to such higher terms that the new denominator will be 8. The rule to follow is:

Rule 3. *To reduce a fraction to higher terms with a given denominator, divide the new denominator by the denominator of the fraction and multiply the numerator and denominator of the fraction by the quotient.*

Following this rule we divide 8 by 2 and the quotient is 4. Then

$$\frac{1\times 4}{2\times 4}=\frac{4}{8}$$

To prove this refer to Fig. 8(b) and (c). In (c) we see that $\frac{1}{4}$ equals two of the $\frac{1}{8}$ parts. So the two $\frac{1}{4}$ parts in (b) are the same size as 4 of the $\frac{1}{8}$ parts in (c). Therefore $\frac{1}{2}$ has the same value as $\frac{4}{8}$.

Note. To make this calculation a little easier to write we can from now on write it

$$\frac{1}{2}=\frac{1\times 4}{2\times 4}=\frac{4}{8}$$

ILLUSTRATIVE EXAMPLES

1. Reduce $\frac{3}{9}$ to twenty-sevenths.

27 (new denominator) $\div$ 9 (old denominator) $=3$ (quotient)

$$\frac{3}{9}=\frac{3\times 3}{9\times 3}=\frac{9}{27}\text{ so }\frac{3}{9}=\frac{9}{27}$$

2. Reduce $\frac{2}{3}$ to thirty-ninths.

$$39\div 3=13\text{, then }\frac{2}{3}=\frac{2\times 13}{3\times 13}=\frac{26}{39}\text{ so }\frac{2}{3}=\frac{26}{39}$$

3. Reduce $\frac{13}{22}$ to sixty-sixths.

$$66\div 22=3\text{, then }\frac{13}{22}=\frac{13\times 3}{22\times 3}=\frac{39}{66}\text{ so }\frac{13}{22}=\frac{39}{66}$$

All of the above examples were worked following **Rule 3.**

PRACTICE PROBLEMS

After you have worked the following problems turn to page 47, where correct answers are shown, and check your answers.

Reduce the following fractions to the given denominator.

1. Reduce $\frac{5}{6}$ to sixtieths.
2. Reduce $\frac{15}{16}$ to sixty-fourths.
3. Reduce $\frac{4}{5}$ to fortieths.
4. Reduce $\frac{9}{25}$ to hundredths.
5. Reduce $\frac{7}{12}$ to forty-eighths.
6. Reduce $\frac{9}{16}$ to eightieths.
7. Reduce $\frac{14}{15}$ to ninetieths.
8. Reduce $\frac{7}{8}$ to fifty-sixths.
9. Reduce $\frac{3}{11}$ to seventy-sevenths.
10. Reduce $\frac{10}{12}$ to seventy-seconds.

Lesson 2

For Step 1, recall Lesson 1. For Step 2, learn the method of reducing a fraction to lower terms. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Reducing Fractions to Lower Terms. To reduce a fraction to lower terms, divide both numerator and denominator by the same number. Here, the procedure is just the reverse of reducing a fraction to higher terms. This process, of course, makes the fraction seem smaller, but its value is not changed. In more advanced mathematics, and even in beginning mathematics, we often want to reduce a fraction to lower terms because it is easier to work with in lower terms. You will see how this works out in the illustrative examples to follow.

Rule 4. *To reduce a fraction to lower terms, divide both the numerator and denominator by the same number.*

Take, for example, the fraction $\frac{3}{9}$. Following the rule we have to divide both the 3 and 9 by the same number in order to reduce $\frac{3}{9}$ to lower terms. In this case we can divide both 3 and 9 by 3.

$$3 \div 3 = 1$$
$$9 \div 3 = 3$$
$$\text{Therefore } \tfrac{3}{9} = \tfrac{1}{3}$$

To prove this refer again to Fig. 9 (*b*). Here we see that the space occupied by three of the $\frac{1}{9}$ parts is exactly the same as the space occupied by the $\frac{1}{3}$ part. Therefore in reducing $\frac{3}{9}$ to lower terms we have not changed its value. Reducing fractions to lower terms does not change their value.

ILLUSTRATIVE EXAMPLES

1. Reduce $\frac{10}{20}$ to lower terms.

By trial we find that 5 will divide an exact number of times into both 10 and 20.

$$\frac{10}{20}=\frac{10\div5}{20\div5}=\frac{2}{4}$$

This answer of $\frac{2}{4}$ satisfies the rule. It can be seen that $\frac{2}{4}$ is much easier to work with than $\frac{10}{20}$ although it has exactly the same value.

There are other ways of solving this problem. The following is also correct.

$$\frac{10}{20}=\frac{10\div2}{20\div2}=\frac{5}{10}$$

Here the $\frac{5}{10}$ is a correct answer too because we have followed the rule.

Also we could solve the problem like this:

$$\frac{10}{20}=\frac{10\div10}{20\div10}=\frac{1}{2}$$

This answer is correct too because we followed the rule.

You can therefore see that in some cases more than one answer is possible in reducing to lower terms. All answers are equally correct and all have the same value. In this example the answers $\frac{2}{4}$, $\frac{5}{10}$, and $\frac{1}{2}$ are all correct and all have the same value as $\frac{10}{20}$.

It is an easy matter to prove your answers. For example we can prove the answer $\frac{1}{2}$ is correct by multiplying both the 1 and 2 by the same number you divided by to obtain them.

$$\text{Thus } \frac{1}{2}=\frac{1\times10}{2\times10}=\frac{10}{20}$$

2. Reduce $\frac{5}{15}$ to lower terms.

By trial we find that both 5 and 15 can be exactly divided by 5.

$$\frac{5}{15}=\frac{5\div5}{15\div5}=\frac{1}{3}$$

In this particular problem no other answer is possible.

3. Reduce $\frac{12}{16}$ to lower terms.

By trial we find that both 12 and 16 can be exactly divided by 2.

$$\frac{12}{16}=\frac{12\div2}{16\div2}=\frac{6}{8}$$

Also 4 can be used as a divisor:

$$\frac{12}{16}=\frac{12\div4}{16\div4}=\frac{3}{4}$$

In Section 2 you learned what the expression "exactly divided by" means and how to find divisor numbers by trial. Review Section 2 if necessary.

Reducing a Fraction to Lower Terms with Given Denominator. Sometimes, because of the nature of calculations in more advanced mathematics, it is necessary to reduce a fraction to lower terms but having a definite denominator.

Rule 5. *To reduce a fraction to lower terms having a required denominator, divide the denominator of the fraction by the required denominator. Then divide the numerator and the denominator of the fraction by this quotient.*

ILLUSTRATIVE EXAMPLES

1. Reduce $\frac{12}{16}$ to fourths.

Following the rule divide 16 by 4 and then divide 12 and 16 by the quotient.

$$16 \div 4 = 4$$

$$\frac{12}{16} = \frac{12 \div 4}{16 \div 4} = \frac{3}{4}$$

2. Reduce $\frac{14}{35}$ to fifths.

$$35 \div 5 = 7$$

$$\frac{14}{35} = \frac{14 \div 7}{35 \div 7} = \frac{2}{5}$$

PRACTICE PROBLEMS

After you have worked the following problems turn to page 47, where correct answers are shown, and check your answers.

1. Reduce $\frac{15}{20}$ to fourths.
2. Reduce $\frac{36}{40}$ to tenths.
3. Reduce $\frac{24}{36}$ to sixths.
4. Reduce $\frac{50}{75}$ to thirds.
5. Reduce $\frac{12}{36}$ to ninths.
6. Reduce $\frac{16}{20}$ to fifths.
7. Reduce $\frac{20}{44}$ to elevenths.
8. Reduce $\frac{30}{45}$ to fifteenths.
9. Reduce $\frac{16}{76}$ to nineteenths.
10. Reduce $\frac{25}{115}$ to twenty-thirds.

Lesson 3

For Step 1, recall the previous lessons. For Step 2, learn the method of reducing fractions to lowest terms. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Reducing a Fraction to the Lowest Possible Terms. This process is much like reducing fractions to lower terms. However, in reducing to lower terms we were not required to reduce to lowest terms.

When we reduce a fraction to its lowest possible terms we reduce it as far as we possibly can, that is, until no further reduction is possible.

You must remember that any problem in mathematics which has a fraction for the answer must have that answer (fraction) reduced to the lowest possible terms.

Rule 6. *To reduce a fraction to its lowest possible terms, use cancellation method.*

You learned the cancellation process in Section 2. Review this section if necessary.

ILLUSTRATIVE EXAMPLES

1. Reduce $\frac{24}{30}$ to lowest terms.

$$\frac{\overset{\overset{4}{\cancel{12}}}{\cancel{24}}}{\underset{\underset{5}{\cancel{15}}}{\cancel{30}}} = \frac{4}{5}$$

Here we followed **Rule 6.** As explained in Section 2, we started out by seeing if 2 would divide both 24 and 30 an exact number of times. It did, so we divided 24 and 30 by 2 and obtained 12 and 15. We find that 2 will not divide both 12 and 15 an exact number of times, so we try the next higher number, 3. The 3 divides into 12 and 15, exactly, 4 and 5 times. No number, above 1, will divide an exact number of times into 4 and 5. Therefore the answer is $\frac{4}{5}$ because it is the lowest possible terms. Remember that $\frac{4}{5}$ has exactly the same value as $\frac{24}{30}$. We could prove this if we had illus-

trations similar to Figs. 8 and 9, with the circles divided into 5ths, 10ths, and 30ths.

2. Reduce $\frac{24}{36}$ to lowest terms.

$$\begin{array}{l} 2 \\ \cancel{6} \\ \cancel{12} \\ \dfrac{\cancel{24}}{\cancel{36}}=\dfrac{2}{3} \\ \cancel{18} \\ \cancel{9} \\ 3 \end{array}$$

Here we divided 24 and 36 by 2.
Then we divided 12 and 18 by 2.
Then we divided 6 and 9 by 3.
No further cancellation is possible so $\frac{2}{3}$ is the answer.

3. Reduce $\frac{64}{80}$ to lowest terms.

$$\begin{array}{l} 4 \\ \cancel{8} \\ \cancel{16} \\ \cancel{32} \\ \dfrac{\cancel{64}}{\cancel{80}}=\dfrac{4}{5} \\ \cancel{40} \\ \cancel{20} \\ \cancel{10} \\ 5 \end{array}$$

Here we used 2 throughout. You can easily follow the calculations.

In this example we could start our cancellation using a larger number if we desired.

$$\begin{array}{l} 4 \\ \cancel{8} \\ \dfrac{\cancel{64}}{\cancel{80}}=\dfrac{4}{5} \\ \cancel{10} \\ 5 \end{array}$$

By trial we found that 8 would exactly divide 64 and 80 just 8 and 10 times. Then we divided 8 and 10 by 2.

Whether you use large or small numbers in cancellation does not make any difference as shown by Example 3. However until you have had considerable experience probably it will be easier for you to use small numbers.

Note: You should actually work out all these illustrative examples for yourself to be sure you understand them.

4. Reduce $\frac{242}{528}$ to lowest terms.

Here we have a fraction made up of large numbers and it is a little difficult to start right out using large numbers for cancellation. Therefore the best policy is to try 2 and see if it will exactly divide into both numerator and denominator. Whenever you encounter a fraction made up of a large numerator and a large denominator it is always best to try one or two very small numbers first.

$$\frac{\overset{\overset{11}{\cancel{121}}}{\cancel{242}}}{\underset{\underset{24}{\cancel{264}}}{\cancel{528}}} = \frac{11}{24}$$

In the above cancellation we found that 2 would exactly divide both 242 and 528. The results were 121 and 264. Next we tried 3, 4, 5, 6, 7, 8, 9, and 10. None of these numbers would exactly divide both 121 and 264. We kept right on trying and found that 11 would exactly divide 121 and 264. The results were 11 and 24. The 11 is the last number we can try because no other number except 1 and 11 will divide an exact number of times into 11.

The lowest possible terms have thus been found and the answer is $\frac{11}{24}$.

PRACTICE PROBLEMS

After you have worked these problems turn to page 47, where correct answers are shown, and check your work.

If you get wrong answers in your solutions review the lesson carefully and keep trying until your answers are correct.

1. Reduce $\frac{121}{165}$ to lowest possible terms.
2. Reduce $\frac{144}{156}$ to lowest possible terms.

3. Reduce $\frac{125}{320}$ to lowest possible terms.
4. Reduce $\frac{1728}{8640}$ to lowest possible terms.
5. Reduce $\frac{55}{121}$ to lowest possible terms.
6. Reduce $\frac{640}{1280}$ to lowest possible terms.
7. Reduce $\frac{120}{960}$ to lowest possible terms.
8. Reduce $\frac{119}{161}$ to lowest possible terms.
9. Reduce $\frac{143}{187}$ to lowest possible terms.
10. Reduce $\frac{2236}{8944}$ to lowest possible terms.

Lesson 4

For Step 1, recall the previous lessons. For Step 2, learn the method of working with improper fractions. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Up to this point in your study of fractions all of the fractions you have worked with have had a denominator larger than the numerator. For example the common fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{16}$, $\frac{55}{121}$, $\frac{80}{160}$, etc., all have denominators larger than their numerators.

A fraction having a denominator which is larger than its numerator is called a **proper fraction.**

Improper Fractions. Now we go another step forward and learn about a fraction which is a little different from those you have seen so far. In your advanced study of mathematics you will often see fractions having numerators larger than their denominators.

A fraction having a numerator which is larger than its denominator is called an **improper fraction.**

You will remember that the denominator of a fraction shows into how many parts a unit has been divided, and that the numerator shows how many of these parts we have in mind. For example, $\frac{3}{10}$ of a cake means that the cake was divided into 10 equal parts and that we are thinking of 3 of these parts.

In improper fractions the same reasoning is followed except that now we are thinking of more parts than any one object actually has been divided into.

Typical improper fractions are, $\frac{7}{5}$, $\frac{8}{3}$, $\frac{4}{3}$, $\frac{10}{4}$, $\frac{87}{15}$, $\frac{21}{7}$, $\frac{8}{7}$, $\frac{15}{4}$, etc.

Take $\frac{7}{5}$ for example. The denominator indicates that a unit has been divided into 5 parts. The 7 indicates that we are thinking of 7 such parts. Now, if one whole cake or the number 1 is divided into 5 parts, then indicating 7 such parts means that we are really thinking

of more than 1 cake or one number 1. In other words, we are thinking of one whole cake, or one whole number 1, plus parts of another cake or another number 1. Therefore using an improper fraction shows that more than one number 1 is meant.

Mixed Numbers. In your mathematics work you will also see many combinations of whole numbers and proper fractions.

A number which is a combination of a whole number and a proper fraction is called a **mixed number.**

If we have one whole cake and someone gives us $\frac{1}{4}$ of another cake it is easy to see that we would have 1 plus $\frac{1}{4}$ or $1\frac{1}{4}$ cakes. Or, if we had one whole number such as 12 and added $\frac{1}{4}$ to it we would have $12\frac{1}{4}$.

Mixed numbers and improper fractions are very closely related and knowing one we can always easily calculate the other.

Rule 7. *To change a mixed number to an improper fraction, multiply the whole number by the denominator of the fraction and then add the numerator of the fraction, which gives the new numerator to put above the denominator.*

ILLUSTRATIVE EXAMPLES

1. Change $6\frac{8}{9}$ to an improper fraction.

Solution. Following **Rule 7** we first multiply the whole number (6) by the denominator (9) of the fraction.

$$6 \times 9 = 54$$

Next we add the numerator (8) of the fraction to 54.

$$54 + 8 = 62$$

The 62 becomes the numerator of the improper fraction and its denominator remains the same as in the original fraction, or 9.

$$\text{Thus } 6\frac{8}{9} = \frac{62}{9}$$

2. Change $25\frac{2}{3}$ to an improper fraction.

Solution. Multiplying the whole number by the denominator of the fraction, gives

$$25 \times 3 = 75$$

Adding the numerator gives

$$75 + 2 = 77$$

The 77 is the numerator of the improper fraction, and the denominator (3) is the original one.

$$\text{Thus } 25\tfrac{2}{3}=\tfrac{77}{3}$$

3. Change $7\frac{3}{8}$ to an improper fraction.

Solution

$$7\tfrac{3}{8}=7\times8=56 \text{ and } 56+3=59 \text{ so } 7\tfrac{3}{8}=\tfrac{59}{8}$$

This is a simpler way of showing exactly how the rule is followed.

4. Change $16\frac{7}{10}$ to an improper fraction.

Solution

$$16\tfrac{7}{10}=\frac{16\times10+7}{10}=\frac{167}{10}$$

This is the easiest way of following **Rule 7** and is the way recommended for you to change mixed numbers to improper fractions. You can see that we followed the rule exactly. We multiplied the whole number by the denominator of the fraction (16×10). We added the numerator of the fraction ($16\times10+7$). This $16\times10+7$ is the numerator, and the denominator (10) remains the same as in the original fraction. If we multiply 16 by 10 we get 160. If we add 7 to 160 we get 167. Thus the answer is $\frac{167}{10}$.

5. Change $30\frac{3}{20}$ to an improper fraction.

Solution

$$30\tfrac{3}{20}=\frac{30\times20+3}{20}=\frac{603}{20}$$

Rule 8. *To change an improper fraction to a mixed number, divide the numerator by the denominator.*

In a fraction the horizontal line between the numerator and the denominator indicates division just the same as $10\div5$ indicates division. Thus $\frac{10}{5}$ means that we can divide 10 (the numerator) by 5 (the denominator). As you have already found, and as you will find in Section 4, we do not always think about this division. But when we want to divide we can, and sometimes it is necessary.

If we have a fraction $\frac{8}{8}$, for example, we know that the object in mind has been divided into 8 equal parts and that we are thinking of 8 such parts. If we are thinking of 8 parts of an object that has been divided into 8 parts we are thinking of the whole object.

Thus $\frac{8}{8}=1$. In like manner $\frac{10}{10}=1$, $\frac{77}{77}=1$, $\frac{100}{100}=1$, etc.

If an object has been divided into 10 parts and we are thinking of 20 parts, we have in mind two objects which have each been divided into 10 parts. Thus

$$\cdot=2$$

In like manner $\frac{50}{5}=10$, $\frac{60}{5}=12$, $\frac{6}{3}=2$, $\frac{100}{10}=10$, $\frac{34}{17}=2$, $\frac{90}{3}=30$, etc. The division is done exactly as explained in Section 1.

When we divide the numerator of a fraction by its denominator, we get either a whole number or a mixed number. The above illustrations show how some fractions give a whole number. Now suppose we have the fraction $\frac{10}{3}$. If we divide the numerator by the denominator we get a quotient of 3 and have a remainder of 1. In such a case this remainder is used as a numerator over the original denominator. Thus

$$\frac{10}{3}=3\tfrac{1}{3}$$

ILLUSTRATIVE EXAMPLES

1. Change $\frac{5}{3}$ to a mixed number.

Solution. Following **Rule** 8 we divide 5 by 3. The 3 divides into 5 once with a remainder of 2. Thus

$$\frac{5}{3}=1\tfrac{2}{3}$$

The quotient (1) is the whole number of the mixed number. The remainder (2) is placed as a numerator over the original denominator.

We can prove that $1\frac{2}{3}$ is correct by applying **Rule 7.**

$$1\tfrac{2}{3}=\frac{1\times 3+2}{3}=\frac{5}{3}$$

2. Change $\frac{485}{12}$ to a mixed number.

Solution. Divide 485 by 12 using the method you learned in Section 1. This gives a quotient of 40 and a remainder of 5. Then

$$\frac{485}{12}=40\tfrac{5}{12}$$

3. Change $\frac{63}{4}$ to a mixed number.

Solution. Dividing 63 by 4 gives a quotient of 15 and a remainder of 3. Thus

$$\frac{63}{4}=15\tfrac{3}{4}$$

4. Change $\frac{75}{20}$ to a mixed number.

Solution. Dividing 75 by 20 gives a quotient of 3 and a remainder of 15. Thus

$$\frac{75}{20}=3\frac{15}{20}=3\frac{3}{4}$$

Here the $\frac{15}{20}$ was reduced to lowest terms.

5. Change $\frac{85}{15}$ to a mixed number.

Solution. Dividing 85 by 15 gives a quotient of 5 and a remainder of 10. Thus

$$\frac{85}{15}=5\frac{10}{15}=5\frac{2}{3}$$

PRACTICE PROBLEMS

After you have worked all of the following problems compare your answers with the correct answers shown on Page 47.

Change the following mixed numbers to improper fractions.

1. $8\frac{3}{7}$
2. $12\frac{5}{9}$
3. $21\frac{8}{11}$
4. $1000\frac{5}{6}$
5. $83\frac{1}{3}$
6. $76\frac{4}{5}$
7. $46\frac{7}{10}$
8. $56\frac{13}{14}$
9. $37\frac{6}{7}$
10. $23\frac{75}{76}$

Change the following improper fractions to mixed numbers.

11. $\frac{8}{5}$
12. $\frac{125}{2}$
13. $\frac{77}{10}$
14. $\frac{201}{11}$
15. $\frac{48}{7}$
16. $\frac{763}{18}$
17. $\frac{200}{3}$
18. $\frac{87}{6}$
19. $\frac{94}{7}$
20. $\frac{86}{4}$

Lesson 5

For Step 1, recall the previous lessons. For Step 2, learn all of the new names. For Step 3, learn how to find least common denominator and how to reduce fractions to L.C.D. For Step 4, work out the Illustrative Examples. For Step 5, work the Practice Problems.

LEAST COMMON DENOMINATOR

So far you have learned what fractions are and how to reduce or change them from one form to another such as reducing to lower terms, reducing to higher terms, and changing improper fractions to mixed numbers. All of these operations have concerned *single* fractions. In other words you worked with only *one* fraction at a time and did not attempt to consider groups of fractions.

Now you will learn what must be done to a group of two or more fractions so they can be added together or subtracted one from another. The actual addition and subtraction of fractions is explained in the next lesson, but the preparations necessary for adding and subtracting are explained here. Therefore learn the following rules, keeping in mind that they are the foundation for the next lesson.

First you must learn a few new names.

Common Denominator. You already know that a denominator is the lower portion of a fraction and that a denominator shows into how many parts a unit has been divided. If we have two or more fractions all having the same denominator, the fractions are said to have a common denominator. The word "common" means that all of the fractions in any one group have the same denominator.

Take for example the following group of fractions.

$$\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{6}{8}, \frac{4}{8}, \frac{5}{8}, \frac{7}{8},$$

These fractions have a **common denominator** because every one of them has 8 for its denominator. Now note the next group of fractions.

$$\frac{4}{10}, \frac{3}{8}, \frac{3}{10}, \frac{6}{7}, \frac{2}{3}$$

This group *does not* have a common denominator because there are four different denominators.

Similar Fractions. Fractions having the same denominator are called similar fractions. You can see that "similar fractions" and "common denominator" are closely related.

Least Common Denominator. The smallest (least) number into which the denominators of a group of two or more fractions will divide an exact number of times is called a least common denominator. To illustrate this take the fractions

$$\frac{1}{8}, \quad , \frac{3}{8}, \frac{6}{8}, \ 4 \quad 5 \quad \frac{7}{8}$$

Here the denominators are all common (the same) and you can see that 8 is the least (smallest) number into which these denominators will divide an exact number of times. Therefore 8 is the least common denominator. Now take the fractions

$$\frac{4}{10}, \frac{3}{8}, \frac{3}{10}, \frac{6}{7}, \frac{2}{3}$$

If we can find the least (smallest) number into which all of these denominators will divide an exact number of times, that number will be the least common denominator.

Finding the Least Common Denominator. The following rule and methods show you how to find the least common denominator for groups of fractions which do not have common denominators. This rule is given in steps to make it easier to understand and use.

Rule 9. *To find the least common denominator for a group of fractions which do not have common denominators:*

Step 1. Place the denominators of the group of fractions in a horizontal line with a space between them. Draw the division sign.

Step 2. By trial find the *smallest* number that will divide an exact number of times into two or more of the denominators.

Step 3. Divide and bring down the quotients under the denominators that were divisible an exact number of times.

Step 4. Bring down the denominators that were not divisible.

Step 5. Divide this second set of numbers by the smallest number that will divide **two** or more of them an exact number of times and proceed as in **Steps 3** and **4.**

Step 6. Continue this operation of writing new lines and dividing by the smallest number that will divide **two** or more of the numbers until you cannot divide any more.

Step 7. Multiply together all the divisors and all numbers left in the last line. This product is the **least common denominator.**

For convenience we will use the initials L.C.D.

ILLUSTRATIVE EXAMPLES

1. Find the L.C.D. (least common denominator) for the fractions $\frac{1}{6}$, $\frac{3}{8}$, $\frac{2}{9}$, $\frac{5}{12}$, $\frac{5}{18}$, $\frac{7}{24}$, $\frac{1}{36}$

The steps shown in this **solution** are numbered as in **Rule 9.**

Instruction	*Operation*

Step 1

First place the denominators as instructed in **Step 1** of **Rule 9.** You will note that we pay no attention to the numerators and write the denominators alone. The division sign is drawn in.

Step 1

$$\begin{array}{r|ccccccc} & 6 & 8 & 9 & 12 & 18 & 24 & 36 \\ \hline \end{array}$$

Step 2

Following **Step 2** of the rule we consider all the denominators and first try 2, which is the smallest number. We find, by trial, that it will divide an exact number of times into not only *two*, but several of the denominators.

Step 2

$$\begin{array}{r|ccccccc} 2 & 6 & 8 & 9 & 12 & 18 & 24 & 36 \\ \hline \end{array}$$

Step 3

Divide 6, 8, 12, 18, 24, and 36 by the 2 and show the quotients under each of the denominators. For example, the 2 divides into 6 exactly 3 times so the 3 is put directly under the 6.

The 9 cannot be divided an exact number of times by 2.

Step 3

$$\begin{array}{r|ccccccc} 2 & 6 & 8 & 9 & 12 & 18 & 24 & 36 \\ \hline & 3 & 4 & & 6 & 9 & 12 & 18 \end{array}$$

Step 4

Following **Step 4** of the rule we bring down the 9 because it was not divided. Put in new division sign.

Step 4

$$\begin{array}{r|ccccccc} 2 & 6 & 8 & 9 & 12 & 18 & 24 & 36 \\ \hline & 3 & 4 & 9 & 6 & 9 & 12 & 18 \\ \hline \end{array}$$

Step 5

By trial we find that 2 can be used again because it divides two or more of the numbers. Following **Steps 3** and **4** we have a new line of numbers.

Step 5

$$\begin{array}{r|ccccccc} 2 & 6 & 8 & 9 & 12 & 18 & 24 & 36 \\ \hline 2 & 3 & 4 & 9 & 6 & 9 & 12 & 18 \\ \hline & 3 & 2 & 9 & 3 & 9 & 6 & 9 \end{array}$$

Step 6

Following **Step 6** of the rule and by trial we find that 2 can be used for the third line. We cannot use 2 for the fourth line so 3 is tried and found satisfactory. The 3 can also be used for the fifth line. The last line is all 1's so no further division is possible.

Step 6

$$\begin{array}{r|rrrrrrr}
2) & 6 & 8 & 9 & 12 & 18 & 24 & 36 \\ \hline
2) & 3 & 4 & 9 & 6 & 9 & 12 & 18 \\ \hline
2) & 3 & 2 & 9 & 3 & 9 & 6 & 9 \\ \hline
3) & 3 & 1 & 9 & 3 & 9 & 3 & 9 \\ \hline
3) & 1 & 1 & 3 & 1 & 3 & 1 & 3 \\ \hline
 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Step 7

Following **Step 7** of the rule we multiply together all the divisors and all the numbers left in the last line, which gives a product of 72. This 72 is the L.C.D.

To be sure that 72 is really the L.C.D. we divide it by all of the denominators of the fractions we began with and find that each of these denominators divides into 72 an exact number of times.

Step 7

$2 \times 2 \times 2 \times 3 \times 3 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 72$

$$\begin{array}{r}
4 \\
\underline{2} \\
8 \\
\underline{3} \\
24 \\
\underline{3} \\
72 \\
\underline{1} \\
72 \\
\underline{1} \\
72 \\
\underline{1} \\
72 \\
\underline{1} \\
72 \\
\underline{1} \\
72 \\
\underline{1} \\
72 \\
\underline{1} \\
72 \text{ L.C.D.}
\end{array}$$

The reason why we find the *least* or smallest common denominator is because smaller numbers are easier to handle than larger numbers.

Note: Do not be satisfied to just read the preceding illustrative example. **Be sure** you can work it yourself without looking at the solution shown in the text. You learn more quickly by actual experience.

2. Find the L.C.D. of $\frac{7}{36}, \frac{3}{25}, \frac{4}{39}, \frac{5}{27}, \frac{9}{44}$.

Solution

$$\begin{array}{r|rrrrr}
2) & 36 & 25 & 39 & 27 & 44 \\ \hline
2) & 18 & 25 & 39 & 27 & 22 \\ \hline
3) & 9 & 25 & 39 & 27 & 11 \\ \hline
3) & 3 & 25 & 13 & 9 & 11 \\ \hline
 & 1 & 25 & 13 & 3 & 11
\end{array}$$

$2 \times 2 \times 3 \times 3 \times 1 \times 25 \times 13 \times 3 \times 11 = 386{,}100$, L.C.D.

Here we followed all 7 steps of **Rule 9**, just as in Example 1. We used 2 twice and 3 twice. After this no further division was possible because we could not find any number that would exactly divide 1, 25, 13, 3, and 11.

The L. C. D. is a very large number. This problem was used so as to show you that with some combinations of denominators the L.C.D. would be very large.

3. Find L.C.D. of $\frac{4}{15}$, $\frac{3}{20}$, $\frac{7}{30}$, $\frac{9}{10}$.

Solution

$$\begin{array}{r|rrrr} 2) & 15 & 20 & 30 & 10 \\ \hline 3) & 15 & 10 & 15 & 5 \\ \hline 5) & 5 & 10 & 5 & 5 \\ \hline & 1 & 2 & 1 & 1 \end{array}$$

$$2\times3\times5\times1\times2\times1\times1=60, \text{ L.C.D.}$$

4. Find L.C.D. of $\frac{2}{5}$, $\frac{1}{7}$, $\frac{4}{11}$, $\frac{6}{13}$, $\frac{2}{3}$.

Solution

$$\begin{array}{r|rrrrr}) & 5 & 7 & 11 & 13 & 3 \\ \hline \end{array}$$

We cannot divide here because we cannot find any number that will exactly divide two or more of the denominators. You will notice that all of the denominators are **prime** numbers. (Review Section 2 for Prime Numbers.)

The only way to find the L.C.D. for a group of prime number denominators is to multiply all the denominators together. The product is then the L.C.D.

$$5\times7\times11\times13\times3=15015, \text{ L.C.D.}$$

It is possible to find the L.C.D. of a group of fraction denominators by starting with a divisor that is not the smallest divisor. However, it is best to follow **Rule 9** in all such problems, because by so doing you will not become confused.

PRACTICE PROBLEMS

After you have found the L.C.D. in the following problems turn to Page 47 and check your answers with the answers shown there.

If your answers do not check with the correct answers review **Rule 9** very carefully and work your problems over and over until you find your mistakes.

1. Find L.C.D. of $\frac{1}{6}$, $\frac{1}{12}$.
2. Find L.C.D. of $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{24}$.
3. Find L.C.D. of $\frac{3}{10}$, $\frac{4}{15}$ $\frac{7}{20}$.

4. Find L.C.D. of $\frac{3}{7}$, $\frac{4}{11}$, $\frac{6}{7}$, $\frac{2}{13}$.
5. Find L.C.D. of $\frac{2}{3}$, $\frac{4}{5}$, $\frac{5}{11}$, $\frac{1}{5}$, $\frac{2}{17}$.
6. Find L.C.D. of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{16}$, $\frac{6}{20}$, $\frac{5}{15}$.
7. Find L.C.D. of $\frac{6}{30}$, $\frac{15}{18}$, $\frac{10}{55}$, $\frac{16}{56}$.
8. Find L.C.D. of $\frac{3}{5}$, $\frac{4}{15}$, $\frac{10}{20}$, $\frac{6}{25}$.
9. Find L.C.D. of $\frac{1}{12}$, $\frac{3}{6}$, $\frac{4}{15}$, $\frac{1}{10}$.
10. Find L.C.D. of $\frac{1}{12}$, $\frac{1}{8}$, $\frac{1}{6}$.

Reducing Fractions to L.C.D. You have learned how to find the L.C.D. for two or more denominators of fractions.

Now you will learn how to reduce (change) complete fractions to their L.C.D. The idea back of reducing groups of fractions to L.C.D. is to change them so they all have the same denominator. In other words, we want to change a group of different fractions so they are all in terms of the same denominator. You have already learned that you can change a fraction to higher or lowest terms without changing its value.

Rule 10. *To reduce (change) fractions to their L.C.D., divide the L.C.D. by each denominator in turn and multiply each term (both numerator and denominator) of the fractions by the result.*

ILLUSTRATIVE EXAMPLES

1. Reduce $\frac{1}{6}$, $\frac{3}{8}$, $\frac{2}{9}$, $\frac{5}{12}$, $\frac{5}{18}$, $\frac{7}{24}$, $\frac{1}{36}$ to their lowest common denominator.

Solution. In the previous group of Illustrative Examples you learned how to find L.C.D. **for the denominators** of a group of fractions. Here we are to change the **complete fractions** so they are expressed in terms of the L.C.D.

In Example 1, of the previous group of Illustrative Examples, we found that the L.C.D. for the denominators of the fractions in this example was 72.

Now, following **Rule 10,** the process is as follows:

Take the first fraction, $\frac{1}{6}$. Divide the L.C.D. by the denominator of the fraction.

$$72 \div 6 = 12$$

Next multiply both numerator and denominator of the $\frac{1}{6}$ by the 12.

$$\frac{1 \times 12}{6 \times 12} = \frac{12}{72}$$

For ease in writing this operation we can write it like this:

$$\frac{1 \times 12}{6 \times 12} = \frac{12}{72}$$

Thus the fraction $\frac{1}{6}$ has been changed so that it is expressed in terms of the L.C.D. for all the fraction denominators. We can prove this by reducing $\frac{12}{72}$ to the lowest terms.

$$\frac{\overset{\overset{\overset{1}{\cancel{3}}}{\cancel{6}}}{\cancel{12}}}{\underset{\underset{\underset{6}{\cancel{18}}}{\cancel{36}}}{\cancel{72}}} = \frac{1}{6}$$

(This was done following **Rule 6**)

The remaining fractions in this problem are handled in the same way. The following is a convenient way of writing the operations.

$$72 \div 8 = 9 \text{ and } \frac{3}{8} = \frac{3 \times 9}{8 \times 9} = \frac{27}{72}$$
$$72 \div 9 = 8 \text{ and } \frac{2}{9} = \frac{2 \times 8}{9 \times 8} = \frac{16}{72}$$
$$72 \div 12 = 6 \text{ and } \frac{5}{12} = \frac{5 \times 6}{12 \times 6} = \frac{30}{72}$$
$$72 \div 18 = 4 \text{ and } \frac{5}{18} = \frac{5 \times 4}{18 \times 4} = \frac{20}{72}$$
$$72 \div 24 = 3 \text{ and } \frac{7}{24} = \frac{7 \times 3}{24 \times 3} = \frac{21}{72}$$
$$72 \div 36 = 2 \text{ and } \frac{1}{36} = \frac{1 \times 2}{36 \times 2} = \frac{2}{72}$$

2. Reduce $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{5}{12}$ to their lowest common denominator.

Solution. In Example 1 we knew what the L.C.D. was from a previous example. In this example we must find it before going ahead with the solution of the example.

Use **Rule 9** to find L.C.D. of all the denominators.

2)	4	6	8	12
2)	2	3	4	6
3)	1	3	2	3
	1	1	2	1

$2 \times 2 \times 3 \times 1 \times 1 \times 2 \times 1 = 24$, L.C.D.

Next use **Rule 10** to reduce (change) the complete fractions to L.C.D. This is done exactly as explained for Example 1.

$$24 \div 4 = 6 \text{ and } \tfrac{3}{4} = \tfrac{3\times 6}{4\times 6} = \tfrac{18}{24}$$
$$24 \div 6 = 4 \text{ and } \tfrac{5}{6} = \tfrac{5\times 4}{6\times 4} = \tfrac{20}{24}$$
$$24 \div 8 = 3 \text{ and } \tfrac{7}{8} = \tfrac{7\times 3}{8\times 3} = \tfrac{21}{24}$$
$$24 \div 12 = 2 \text{ and } \tfrac{5}{12} = \tfrac{5\times 2}{12\times 2} = \tfrac{10}{24}$$

3. Reduce $\frac{7}{8}, \frac{9}{10}, \frac{15}{16}, \frac{24}{26}, \frac{9}{15}, \frac{6}{39}$ to their L.C.D.

Solution

First use **Rule 9.**

2)	8	10	16	26	15	39
2)	4	5	8	13	15	39
2)	2	5	4	13	15	39
3)	1	5	2	13	15	39
5)	1	5	2	13	5	13
13)	1	1	2	13	1	13
	1	1	2	1	1	1

$2\times 2\times 2\times 3\times 5\times 13\times 1\times 1\times 2\times 1\times 1\times 1 = 3120$, L.C.D.

Next use **Rule 10.**

$$3120 \div 8 = 390 \text{ and } \tfrac{7}{8} = \tfrac{7\times 390}{8\times 390} = \tfrac{2730}{3120}$$
$$3120 \div 10 = 312 \text{ and } \tfrac{9}{10} = \tfrac{9\times 312}{10\times 312} = \tfrac{2808}{3120}$$
$$3120 \div 16 = 195 \text{ and } \tfrac{15}{16} = \tfrac{15\times 195}{16\times 195} = \tfrac{2925}{3120}$$
$$3120 \div 26 = 120 \text{ and } \tfrac{24}{26} = \tfrac{24\times 120}{26\times 120} = \tfrac{2880}{3120}$$
$$3120 \div 15 = 208 \text{ and } \tfrac{9}{15} = \tfrac{9\times 208}{15\times 208} = \tfrac{1872}{3120}$$
$$3120 \div 39 = 80 \text{ and } \tfrac{6}{39} = \tfrac{6\times 80}{39\times 80} = \tfrac{480}{3120}$$

This solution involves large numbers and requires very careful calculating to avoid errors. You will often encounter such examples.

PRACTICE PROBLEMS

After you have worked the following problems compare your answers with the answers shown on Page 47.

Reduce the following fractions to L.C.D

1. $\frac{1}{7}, \frac{3}{8}, \frac{1}{4}, \frac{2}{3}$
2. $\frac{2}{7}, \frac{1}{3}, \frac{3}{4}, \frac{1}{2}$
3. $\frac{1}{3}, \frac{2}{9}, \frac{5}{12}, \frac{3}{8}$
4. $\frac{1}{15}, \frac{3}{10}, \frac{4}{25}, \frac{1}{30}$
5. $\frac{1}{3}, \frac{2}{7}, \frac{4}{11}$

Lesson 6

For Step 1, recall Lesson 5 and addition of numbers. For Step 2, learn the method of addition of fractions. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

ADDITION OF FRACTIONS

In your study of Section 1 you learned that it was impossible to add oranges to apples, fence posts to dollars, dollars to cents, etc. In other words only quantities of like kind can be added to make one sum. To demonstrate this we will study a typical example.

Suppose you had $3.00, in dollar bills, and 400 pennies and that you wanted to add these two quantities of money to see what the *sum* was. How would you do it?

It is an easy matter to add the one dollar bills simply by counting them. But we cannot add dollars to pennies because they are entirely different in name and value per unit. In other words, a dollar bill is vastly different in value from a penny.

From your experience with money you know that you would change the pennies into dollars and then add the dollars. You know that 400 pennies equals $4.00. Then it is an easy matter to add the $4.00 to the $3.00 and find that the sum is $7.00.

How would we change 400 pennies to dollars? We say that we know a dollar contains 100 pennies so we would divide the 400 pennies by 100 to find the number of dollars. What did we do in terms of mathematics? We changed (reduced) one of the quantities to terms of the other. This made it possible to add the two quantities. In mathematics we call this "reducing the two quantities to a common denominator," which simply means that we found a common term in which we could express the amounts of both quantities.

As a further example of reducing unlike objects to common terms in order to add them we will assume that you have 6 apples and 10 oranges. You cannot add apples and oranges because they are entirely different objects. Therefore you must seek some means of reducing both to a common term.

If you think of the term "fruit" you see that both apples and oranges can be described by this *term*. In other words, 6 apples can be called 6 pieces of *fruit* and 10 oranges can be called 10 pieces

of *fruit*. Thus by reducing apples and oranges to the common term "fruit" you can add them and have a sum of 16 pieces of fruit.

We can carry this same reasoning into our study of the addition of fractions. A few examples will illustrate what is meant.

Suppose you had 2 halves of an apple. You know that you can put these halves together and have a complete or whole apple. Or,

$$\frac{1}{2}+\frac{1}{2}=1$$

Next suppose you had one half ($\frac{1}{2}$) of an apple and one third ($\frac{1}{3}$) of another apple, both apples alike in size. If you put these two sections ($\frac{1}{2}$ and $\frac{1}{3}$) together what part of a whole apple would you have? You cannot tell because you cannot add $\frac{1}{2}+\frac{1}{3}$ as you did $\frac{1}{2}+\frac{1}{2}$.

Now you must learn a new step in your study of fractions because even though the $\frac{1}{2}$ apple and $\frac{1}{3}$ apple are both expressed in the common term "apple" we still cannot add them because as shown above we cannot add $\frac{1}{2}$ to $\frac{1}{3}$. The reason we cannot add $\frac{1}{2}$ to $\frac{1}{3}$ is because the denominator of the $\frac{1}{2}$ indicates the apple was divided into 2 parts and the denominator of the $\frac{1}{3}$ indicates the apple was divided into 3 parts. *Before these parts can be added they must all be the same size or each of the same value.*

Ordinarily, when we add fractions we do not have to think whether they are all parts of apples, etc., or not. *But we do have to remember that we cannot add fractions unless they have the same denominators.* Or, in other words, the denominators of all fractions, in any group being added, must indicate that the objects which they represent have all been divided into the same number of parts.

Rule 11. *To add any group of fractions in which all have the same denominators, the adding process consists of adding the numerators.*

ILLUSTRATIVE EXAMPLES

1. Add $\frac{1}{4}+\frac{2}{4}+\frac{3}{4}$.

Solution. To add these three fractions (all of which have the same denominator) all we need do is add the numerators. Thus $1+2+3=6$. Then

$$\frac{1}{4}+\frac{2}{4}+\frac{3}{4}=\frac{6}{4}$$

The $\frac{6}{4}$ is an improper fraction so it should be changed to a mixed number.

$$\frac{6}{4}=6\div 4=1\frac{2}{4} \text{ (See Lesson 4)}$$

The $\frac{2}{4}$ should be reduced to lowest terms.

$$\frac{\overset{1}{\cancel{2}}}{\underset{2}{\cancel{4}}}=\frac{1}{2} \text{ (See Lesson 3)}$$

Therefore the answer is $1\frac{1}{2}$.

2. Add $\frac{1}{16}+\frac{3}{16}+\frac{9}{16}+\frac{7}{16}+\frac{11}{16}$.

Solution

Adding numerators

$$1+3+9+7+11=31$$

Then the answer $=\frac{31}{16}=1\frac{15}{16}$

Fractions Having Unlike Denominators. You learned previously that we cannot add fractions having unlike denominators. Therefore we must reduce (change) these fractions so that they are all in terms of some common denominator.

In the beginning of Lesson 5, in this Section, the text explained that in that lesson you would learn what must be done to a group of fractions so they could be added. Then in that lesson you learned how to find L.C.D. Also in Lesson 5 you learned how to reduce groups of fractions to their lowest common denominator.

Turn to Lesson 5, under the heading "Reducing Fractions to L.C.D.," and note Illustrative Example 2. In this example you were given $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{5}{12}$ and shown how to reduce them to L.C.D. The results were:

$$\frac{3}{4}=\frac{18}{24} \qquad \frac{5}{6}=\frac{20}{24} \qquad \frac{7}{8}=\frac{21}{24} \qquad \frac{5}{12}=\frac{10}{24}$$

The $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{5}{12}$ were thus reduced to common terms. In other words the fractions were all changed so that they all have the same denominators.

As the denominators are all the same, the fractions can now be added.

From here on the adding is done following **Rule 11.**

$$\frac{18}{24}+\frac{20}{24}+\frac{21}{24}+\frac{10}{24}=18+20+21+10=\frac{69}{24}=2\frac{21}{24}=2\frac{7}{8}$$

The $2\frac{7}{8}$ is the answer obtained by adding $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{5}{12}$.

ILLUSTRATIVE EXAMPLES

1. Add $\frac{2}{3}+\frac{7}{18}+\frac{20}{26}+\frac{27}{21}$.

Solution

Instruction	*Operation*

Step 1

Use **Rule 9.** This step is done as explained for the Illustrative Examples which follow **Rule 9.**

Step 1

$$\begin{array}{r|rrrr} 2 & 3 & 18 & 26 & 21 \\ \hline 3 & 3 & 9 & 13 & 21 \\ \hline & 1 & 3 & 13 & 7 \end{array}$$

Step 2

Multiply all divisors and numbers left in the last line of **Step 1.** This follows **Step 7** of **Rule 9.**

Step 2

$2\times3\times1\times3\times13\times7=1638=$ L.C.D.

Step 3

Use **Rule 10.** This step is carried out as explained for the Illustrative Problems which follow **Rule 10.**

Step 3

$1638\div3=546$ and $\frac{2}{3}=\frac{2\times546}{3\times546}=\frac{1092}{1638}$

$1638\div18=91$ and $\frac{7}{18}=\frac{7\times91}{18\times91}=\frac{637}{1638}$

$1638\div26=63$ and $\frac{20}{26}=\frac{20\times63}{26\times63}=\frac{1260}{1638}$

$1638\div21=78$ and $\frac{27}{21}=\frac{27\times78}{21\times78}=\frac{2106}{1638}$

Step 4

Add the numerators and put their sum as new numerator over 1638 (L.C.D.).

Step 4

$1092+637+1260+2106=5095$

Answer $=\frac{5095}{1638}$

Step 5

Whenever any answer is an improper fraction it must be reduced to a mixed number. Use **Rule 8.**

Step 5

$5095\div1638=3\frac{181}{1638}$ Ans.

This answer is also in lowest terms.

In the above example we could have reduced the original fractions to lowest terms before finding L.C.D. This would have simplified the problem somewhat. For example the $\frac{20}{26}$ and $\frac{27}{21}$ could have been reduced to $\frac{10}{13}$ and $\frac{9}{7}$ by the process explained in Lesson 3. Then the problem would have been to add $\frac{2}{3}+\frac{7}{18}+\frac{10}{13}+\frac{9}{7}$

PRACTICE PROBLEMS

After you have worked the following problems compare your answers with the correct answers shown on Page 47.

Remember to reduce all answers which are improper fractions to mixed numbers and all fractions to lowest terms. Add the following:

1. $\frac{1}{5}+\frac{1}{4}+\frac{1}{6}$	**5.** $\_\_+\frac{5}{6}+\frac{7}{8}$	**9.** $\frac{7}{8}+\frac{2}{5}+\frac{3}{10}$
2. $\frac{1}{2}+\frac{5}{8}+\frac{9}{10}$	**6.** $\frac{2}{9}+\frac{11}{15}+\frac{4}{5}$	**10.** $\frac{5}{12}+\frac{4}{5}+\frac{2}{3}$
3. $\frac{3}{2}+\frac{7}{9}+\frac{2}{3}$	**7.** $\frac{1}{3}+\frac{3}{8}+\frac{4}{5}$	**11.** $\frac{4}{7}+\frac{1}{3}+\frac{7}{9}$
4. $\frac{31}{24}+\frac{11}{12}+\frac{3}{8}$	**8.** $\frac{3}{4}+\frac{2}{7}+$	**12.** $\frac{15}{28}+\frac{3}{4}+\frac{4}{7}$

Lesson 7

For **Step 1**, think over Lesson 6 and all preceding lessons to be sure you remember everything. For Step 2, learn the method of adding mixed numbers. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

ADDITION OF MIXED NUMBERS

Up to this point you have learned how to add proper and improper fractions. The next step is to learn how to add mixed numbers. A mixed number consists of a whole number and a fraction. To add mixed numbers we add the whole numbers separately as you learned in Section 1. Then the fractions are added as explained in Lesson 6. If the sum of the fractions is an improper fraction, reduce it to a mixed number and add the whole number part of the mixed number to the sum of the whole numbers.

ILLUSTRATIVE EXAMPLES

1. Add $3\frac{1}{3}+2\frac{1}{2}+6\frac{1}{6}$.

Instruction

Step 1

Add the whole numbers and put their sum to one side for the time being.

Step 2

Find the L.C.D. for the fractions. Follow Rule 9.

Operation

Step 1

$3+2+6=11$

Step 2

$$\begin{array}{r|lll} 3) & 3 & 2 & 6 \\ \hline 2) & 1 & 2 & 2 \\ & 1 & 1 & 1 \end{array}$$

$3\times2\times1\times1\times1=6$, L.C.D.

Step 3

Reduce fractions to L.C.D. Follow **Rule 10.**

Step 3

$6 \div 3 = 2$ and $\frac{1}{3} = \frac{1 \times 2}{3 \times 2} = \frac{2}{6}$

$6 \div 2 = 3$ and $\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$

$6 \div 6 = 1$ and $\frac{1}{6} = \frac{1 \times 1}{6 \times 1} = \frac{1}{6}$

Step 4

Add the fractions. This is explained in Lesson 6. This is done by adding the numerators and putting their sum as a new numerator over the L.C.D. Remember that $\frac{6}{6} = 1$. This is explained in Lesson 6.

Step 4

$$\frac{2}{6} + \frac{3}{6} + \frac{1}{6} = 2 + 3 + 1 = \frac{6}{6}$$

$$\frac{6}{6} = 1$$

Step 5

Add the sum of the fractions to the sum of the whole numbers.

Step 5

$11 + 1 = 12$ Answer.

2. Add $7\frac{2}{9} + \frac{27}{10} + 11\frac{2}{7} + \frac{67}{8}$.

Here is a slightly different example in that it contains mixed numbers and improper fractions. The solution is explained as follows:

Solution

Instruction

Operation

Step 1

Reduce the two improper fractions to mixed numbers so that all parts of the example will be in terms of mixed numbers. This operation is done as explained in **Rule 8.**

Step 1

$$\frac{27}{10} = 27 \div 10 = 2\frac{7}{10}$$

$$\frac{67}{8} = 67 \div 8 = 8\frac{3}{8}$$

Step 2

Add the whole numbers as explained in Section 1.

Step 2

$7 + 2 + 11 + 8 = 28$

Step 3

Find L.C.D.

Follow **Rule 9.**

Step 3

$$\begin{array}{r|rrrr} 2 & 9 & 10 & 7 & 8 \\ \hline & 9 & 5 & 7 & 4 \end{array}$$

Step 4

Multiply divisors and last line as explained in **Step 7** of **Rule 9.**

Step 4

$2 \times 9 \times 5 \times 7 \times 4 = 2520$, L.C.D.

Step 5

Reduce fractions to L.C.D. Follow **Rule 10.**

$$2520 \div 9 = 280 \text{ and } \frac{2}{9} = \frac{2 \times 280}{9 \times 280} = \frac{560}{2520}$$
$$2520 \div 10 = 252 \text{ and } \frac{7}{10} = \frac{7 \times 252}{10 \times 252} = \frac{1764}{2520}$$
$$2520 \div 7 = 360 \text{ and } \frac{2}{7} = \frac{2 \times 360}{7 \times 360} = \frac{720}{2520}$$
$$2520 \div 8 = 315 \text{ and } \frac{3}{8} = \frac{3 \times 315}{8 \times 315} = \frac{945}{2520}$$

Step 6

Add the numerators and put their sum as a new numerator over the L.C.D. (This is explained in Lesson 6.)

$$\frac{560}{2520} + \frac{1764}{2520} + \frac{720}{2520} + \frac{945}{2520} =$$
$$560 + 1764 + 720 + 945 = \frac{3989}{2520}$$

Step 7

When an answer is an improper fraction it must be changed to a mixed number. This is explained in **Rule 8.**

$$\frac{3989}{2520} = 3989 \div 2520 = 1\frac{1469}{2520}$$

Step 8

Add the result of **Step 2** to result of **Step 7.**

$$28 + 1\frac{1469}{2520} = 29\frac{1469}{2520} \text{ Ans.}$$

Four important points in this last solution, which should be carefully noted, are:

(1) Reduce improper fractions to mixed numbers.

(2) Add the whole numbers apart from the fractions.

(3) If the sum of the fractions is an improper fraction, reduce it to a mixed number.

(4) Add the sum of the whole numbers to the sum of the fractions.

The following example is presented in a form whereby you must read the statement of the example and decide what you are required to do. These "written problems" are not difficult if you think them over carefully.

3. A coal yard has five coal bins with the following amounts of coal in each: $15\frac{2}{3}$ tons, $22\frac{11}{30}$ tons, $19\frac{7}{20}$ tons, $24\frac{5}{12}$ tons, and $13\frac{4}{9}$ tons. How many tons in the five bins?

Solution

Upon reading the example you can easily see that all you have to do is add the contents of all 5 bins in order to find the amount or sum of all the coal.

Instruction	*Operation*
Step 1 Add the whole numbers separately.	**Step 1** $15+22+19+24+13=93$
Step 2 Find the L.C.D. for the fractions using **Rule 9.**	**Step 2** $\begin{array}{r\|rrrrr} 2) & 3 & 30 & 20 & 42 & 9 \\ \hline 3) & 3 & 15 & 10 & 21 & 9 \\ \hline 5) & 1 & 5 & 10 & 7 & 3 \\ \hline & 1 & 1 & 2 & 7 & 3 \end{array}$
Step 3 Multiply divisors and last line as explained in **Step 7** of **Rule 9.**	**Step 3** $3\times2\times5\times1\times1\times2\times7\times3=1260$, L.C.D.
Step 4 Reduce fractions to L.C.D. Follow **Rule 10.**	**Step 4** $1260\div3=420$ and $\frac{2}{3}=\frac{2\times420}{3\times420}=$ [illegible] $1260\div30=42$ and $\frac{11}{30}=\frac{11\times42}{30\times42}\ \frac{462}{1260}$ $1260\div20=63$ and $\frac{7}{20}=\frac{7\times63}{20\times63}=$ [illegible] $1260\div42=30$ and $\frac{5}{42}=\frac{5\times30}{42\times30}=\frac{150}{1260}$ $1260\div9=140$ and $\frac{4}{9}=\frac{4\times140}{9\times140}=\frac{560}{1260}$
Step 5 Add the numerators and put over L.C.D. as explained in Lesson 6.	**Step 5** $\frac{840}{1260}+\frac{462}{1260}+\frac{441}{1260}+\frac{150}{1260}+\frac{560}{1260}$ $=840+462+441+150+560=\frac{2453}{1260}$
Step 6 Reduce sum of fractions to a mixed number.	**Step 6** $\frac{2453}{1260}=2453\div1260=1\frac{1193}{1260}$
Step 7 Combine the results of **Step 1** and **Step 6.**	**Step 7** $93+1\frac{1193}{1260}=94\frac{1193}{1260}$ Tons. Ans.

PRACTICE PROBLEMS

After you have worked the following problems compare your answers with the correct answers shown on Page 47.

Add the following:

1. $2\frac{5}{8}+3\frac{7}{12}+5\frac{11}{24}$
2. $1\frac{3}{20}+2\frac{7}{12}+3\frac{5}{15}$
3. $6\frac{2}{3}+2\frac{2}{7}+4\frac{4}{15}$
4. $3\frac{5}{12}+\frac{7}{12}+2\frac{9}{24}$
5. $5\frac{1}{12}+\frac{3}{4}+6\frac{2}{3}$
6. $9\frac{2}{3}+7\frac{7}{8}+12\frac{3}{4}$
7. $8\frac{4}{5}+\frac{6}{7}+4\frac{5}{8}$
8. $6\frac{2}{7}+2\frac{1}{3}+5\frac{3}{4}$
9. $3\frac{4}{9}+\frac{1}{2}+15\frac{3}{5}$
10. $4\frac{7}{8}+\frac{7}{18}+20\frac{8}{9}$
11. $4\frac{11}{16}+\frac{7}{24}+\frac{9}{32}$
12. $7\frac{1}{2}+5\frac{2}{3}+10\frac{3}{4}$

Lesson 8

For Step 1, recall subtraction of numbers and Lesson 7. For Step 2, learn the method of subtraction of fractions. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

SUBTRACTION OF FRACTIONS

In Lesson 6 you learned that in order to add two or more fractions their denominators must all be the same. You also learned how to reduce unlike fractions to L.C.D. so that all denominators would be the same.

Before one fraction can be *subtracted* from another fraction the same conditions are necessary as for addition. In other words all fractions must be in terms of a common denominator before one can be subtracted from another. Reducing to L.C.D. is done exactly as explained for addition.

Rule 12. *To subtract fractions having the same denominator, the subtraction process consists of subtracting one numerator from another.*

ILLUSTRATIVE EXAMPLES

1. Subtract $\frac{2}{8}$ from $\frac{7}{8}$.

Solution. This example is written

$$\frac{7}{8}-\frac{2}{8}$$

You learned in Section 1 that in subtraction the minuend is written first, then the minus sign (—), and finally the subtrahend.

Following **Rule 12** we subtract numerator 2 from numerator 7 and obtain a difference or remainder of 5. Thus $\frac{7}{8}-\frac{2}{8}=\frac{5}{8}$.

2. Subtract $\frac{5}{24}$ from $\frac{23}{24}$.

Solution. Following **Rule 12** and the method explained in Example 1, the solution is

$$\frac{23}{24}-\frac{5}{24}=\frac{\overset{\overset{3}{\cancel{9}}}{\cancel{18}}}{\underset{\underset{4}{\cancel{12}}}{\cancel{24}}}=\frac{3}{4}$$

Here the remainder is $\frac{18}{24}$. *However,* whenever an answer is in the form of a fraction it should be shown in lowest terms. To reduce $\frac{18}{24}$ to lowest terms we divided 18 and 24 by 2. Then we divided 9 and 12 by 3. Thus the proper answer is $\frac{3}{4}$.

3. Subtract $\frac{5}{45}$ from $\frac{41}{45}$.

Solution

$$\frac{41}{45}-\frac{5}{45}=\frac{36}{45}=\frac{4}{5} \text{ Ans.}$$

Rule 13. *To subtract fractions having unlike denominators, reduce the fractions to L.C.D. and subtract one numerator from the other.*

ILLUSTRATIVE EXAMPLES

1. Subtract $\frac{7}{12}$ from $\frac{23}{36}$.

Solution. Here the denominators are not the same, so we must reduce them to L.C.D. This is done following **Rules 9** and **10** in Lesson 5. Finding L.C.D., use **Rule 9.**

$$\begin{array}{r|rr} 2) & 12 & 36 \\ \hline 2) & 6 & 18 \\ \hline 3) & 3 & 9 \\ \hline & 1 & 3 \end{array}$$

$$2\times2\times3\times1\times3=36, \text{ L.C.D.}$$

Reducing fractions to L.C.D. use **Rule 10.**

$$36\div12=3 \text{ and } \frac{7}{12}=\frac{7\times3}{12\times3}=\frac{21}{36}$$
$$36\div36=1 \text{ and } \frac{23}{36}=\frac{23\times1}{36\times1}=\frac{23}{36}$$

Now we have reduced $\frac{7}{12}$ to $\frac{21}{36}$ and $\frac{23}{36}$ remains $\frac{23}{36}$. From here on we follow **Rule 12.**

$$\frac{23}{36}-\frac{21}{36}=\frac{2}{36}=\frac{1}{18}$$

Note: As you get more practice in mathematics you will be able to see at a glance for example that the L.C.D. for $\frac{7}{12}$ and $\frac{23}{36}$ is 36. However, it is always best actually to go through the process of **Rule 9** if you are in doubt.

2. Subtract $\frac{7}{11}$ from $\frac{29}{43}$.

Solution. Here the denominators are *prime* numbers. **Rule 9** does not work where all denominators are prime numbers so, as explained in Lesson 5, to obtain the L.C.D. we multiply the denominators together.

$$11\times43=473,\ \text{L.C.D.}$$

Then,

$$473\div11=43 \text{ and } \frac{7}{11}=\frac{7\times43}{11\times43}=\frac{301}{473}$$

$$473\div43=11 \text{ and } \frac{29}{43}=\frac{29\times11}{43\times11}=\frac{319}{473}$$

$$\frac{319}{473}-\frac{301}{473}: \frac{18}{473} \text{ Ans.}$$

3. Subtract $\frac{3}{5}$ from $\frac{7}{8}$.

Solution. In this example we cannot apply **Rule 9** because no number will divide exactly into both 5 and 8. Therefore L.C.D. = $5\times8 = 40$.

$$40\div5=8 \text{ and } \frac{3}{5}=\frac{3\times8}{5\times8}=\frac{24}{40}$$

$$40\div8=5 \text{ and } \frac{7}{8}=\frac{7\times5}{8\times5}=\frac{35}{40}$$

$$\frac{35}{40}-\frac{24}{40}=\frac{11}{40} \text{ Ans.}$$

4. Subtract $\frac{3}{4}$ from $\frac{24}{10}$.

$$\text{L.C.D.} = 20$$

$$20\div4=5 \text{ and } \frac{3}{4}=\frac{3\times5}{4\times5}=\frac{15}{20}$$

$$20\div10=2 \text{ and } \frac{24}{10}=\frac{24\times2}{10\times2}=\frac{48}{20}$$

$$\frac{48}{20}-\frac{15}{20}=\frac{33}{20}=1\frac{13}{20}$$

In the above example we could have reduced the $\frac{24}{10}$ to lower terms before finding L.C.D. This would have made the solution somewhat easier. The $\frac{24}{10}$ reduced to lower terms is

$$\frac{\overset{12}{\cancel{24}}}{\underset{5}{\cancel{10}}}=\frac{12}{5}$$

Under some conditions we would change $\frac{12}{5}$ to a mixed number, but in subtracting we would only have to change it back to an improper fraction.

PRACTICE PROBLEMS

After you have worked the following problems compare your answers with the correct answers shown on Page 47.

1. Subtract $\frac{5}{9}$ from $\frac{8}{9}$.
2. Subtract $\frac{11}{12}$ from $\frac{14}{12}$.
3. Subtract $\frac{6}{27}$ from $\frac{20}{27}$.
4. Subtract $\frac{36}{70}$ from $\frac{49}{70}$.
5. Subtract $\frac{2}{9}$ from $\frac{1}{2}$.
6. Subtract $\frac{4}{17}$ from $\frac{3}{8}$.
7. Subtract $\frac{4}{35}$ from $\frac{84}{120}$.
8. Subtract $\frac{4}{5}$ from $\frac{6}{7}$.
9. Subtract $\frac{5}{12}$ from $\frac{5}{6}$.
10. Subtract $\frac{2}{14}$ from $\frac{2}{7}$.

Lesson 9

For Step 1, recall Lesson 7. For Step 2, learn how to subtract mixed numbers. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

SUBTRACTION OF MIXED NUMBERS

Rule 14. *To subtract mixed numbers we first subtract the fractions, as explained in* **Rules 12** *and* **13** *of Lesson 8, then subtract the whole numbers, and finally add the remainders from the fractions and whole numbers.*

ILLUSTRATIVE EXAMPLES

1. Subtract $4\frac{1}{2}$ from $10\frac{2}{3}$.

Solution. First reduce fractions to L.C.D., following **Rules 9** and **10.**

$$\text{L.C.D.} = 6$$
$$6 \div 2 = 3 \text{ and } \frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$$
$$6 \div 3 = 2 \text{ and } \frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6}$$

Next subtract the two fractions in the proper manner, keeping in mind that the $\frac{4}{6}$ is the minuend.

$$\frac{4}{6} - \frac{3}{6} = \frac{1}{6}$$

Next subtract the whole numbers.

$$10 - 4 = 6$$

Finally add the $\frac{1}{6}$ and 6, which gives an answer of $6\frac{1}{6}$.

We could have solved the example by changing the mixed numbers to improper fractions and then subtracting as for fractions.

$$4\tfrac{1}{2}=\tfrac{9}{2}$$
$$10\tfrac{2}{3}=\tfrac{32}{3}$$
$$\text{L.C.D.}=6$$
$$6\div 2=3 \text{ and } \tfrac{9}{2}=\tfrac{9\times 3}{2\times 3}=\tfrac{27}{6}$$
$$6\div 3=2 \text{ and } \tfrac{32}{3}=\tfrac{32\times 2}{3\times 2}=\tfrac{64}{6}$$

Then $\frac{64}{6}-\frac{27}{6}=\frac{37}{6}=6\frac{1}{6}$ Ans.

The first method of solving the example is recommended especially when large mixed numbers are encountered.

2. Subtract $41\frac{6}{11}$ from $98\frac{3}{7}$.

Solution. First reduce fractions to L.C.D., following **Rules 9** and **10.** Also review Illustrative Example 4 in Lesson 5.

$$\text{L.C.D.} = 77$$
$$77\div 11=\ 7 \text{ and } \tfrac{6}{11}=\tfrac{6\times 7}{11\times 7}=\tfrac{42}{77}$$
$$77\div\ 7=11 \text{ and } \tfrac{3}{7}=\tfrac{3\times 11}{7\times 11}=\tfrac{33}{77}$$

The $\frac{33}{77}$ is the minuend and the $\frac{42}{77}$ is the subtrahend. Here we have a case where, according to the example, it is necessary to subtract $\frac{42}{77}$ from $\frac{33}{77}$. Following **Rule 12** this means subtract 42 from 33. However, we cannot subtract 42 from 33 because the minuend must be larger than the subtrahend before subtraction can be carried on. Therefore we must learn another new method which will teach us what to do in such cases.

The numerator of the minuend fraction is too small. It must be increased in size or value. The process of increasing its size or value is explained as follows.

First we will consider the minuend mixed number alone in order to explain the process.

$$\begin{array}{cc} 97 & 110 \\ \cancel{98} & \dfrac{\cancel{33}}{77} \end{array}$$

In Section 1 you learned that if a minuend number was too small you took 1 from the next number to the left and added it to the number which was too small. Here the minuend of the fraction is too small so we take 1 from the 98. Taking 1 from 98 leaves 97 as shown above. Next we want to add this 1 to the $\frac{33}{77}$. Adding 1 and $\frac{33}{77}$ gives $1\frac{33}{77}$. We still cannot subtract $\frac{42}{77}$ from $1\frac{33}{77}$. Therefore we must change $1\frac{33}{77}$ to an improper fraction, using **Rule 7.**

$$1\tfrac{33}{77}=\tfrac{1\times 77+33}{77}=\tfrac{110}{77}$$

In the above operation we have reduced 1 to terms of the $\frac{33}{77}$ and added it to $\frac{33}{77}$ all at the same time.

Now we can subtract the fractions.

$$\frac{110}{77}-\frac{42}{77}=\frac{68}{77}$$

Then following **Rule 14** we next subtract the whole numbers. Remember that the 98 was reduced to 97.

$$97-41=56$$

Finally, add the remainders from the whole numbers and fractions together.

$$56+\frac{68}{77}=56\frac{68}{77} \text{ Ans.}$$

3. A steel rod measures exactly $15\frac{1}{4}$ feet long. One piece $2\frac{1}{2}$ feet long, another piece $4\frac{1}{6}$ feet long were cut off. Neglecting the waste caused by sawing how much was left of the original rod?

Solution. After reading over this example two or three times and thinking about it, we can see that two pieces of the $15\frac{1}{4}$-foot rod have been cut off, which naturally reduces its length. Also we can reason that if we add the two pieces and subtract their sum from $15\frac{1}{4}$ we will have a remainder which represents the length of the rod after the two pieces have been cut off.

So we will add the $2\frac{1}{2}$ and $4\frac{1}{6}$ first. Follow explanation in Lesson 7.

$$2+4=6$$

$$\text{L.C.D.}=6 \ (\textbf{Rule 9})$$

$$6\div 2=3 \text{ and } \frac{1}{2}=\frac{1\times 3}{2\times 3}=$$

$$6\div 6=1 \text{ and } \frac{1}{6}=\frac{1\times 1}{6\times 1}= \qquad (\textbf{Rule 10})$$

Then $\frac{3}{6}+\frac{1}{6}=\frac{4}{6}$

$$\frac{\overset{2}{\cancel{4}}}{\underset{3}{\cancel{6}}}=\frac{2}{3}$$

Finally, $6+\frac{2}{3}=6\frac{2}{3}$ sum of the two pieces which were cut off the bar.

Next we must subtract $6\frac{2}{3}$ from $15\frac{1}{4}$. Follow **Rule 14.**

$$\text{L.C.D.}=12 \qquad (\textbf{Rule 9})$$

$$12\div 4=3 \text{ and } \frac{1}{4}=\frac{1\times 3}{4\times 3}=\frac{3}{12} \qquad (\textbf{Rule 10})$$

$$12\div 3=4 \text{ and } \frac{2}{3}=\frac{2\times 4}{3\times 4}=\frac{8}{12}$$

Here we cannot subtract $\frac{8}{12}$ from $\frac{3}{12}$ so we must take 1 from the 15 which leaves it 14. Then $1+\frac{3}{12}=1\frac{3}{12}$

$$1\frac{3}{12}=\frac{1\times12+3}{12}=\frac{15}{12} \qquad \text{(Rule 7)}$$

Subtracting the fractions,

$$\frac{15}{12}-\frac{8}{12}=\frac{7}{12}$$

Subtracting whole numbers,

$$14-6=8$$

Adding remainders from whole numbers and fractions.

$8+\frac{7}{12}=8\frac{7}{12}$ Ans. This is the amount of bar left after the two pieces were cut off.

4. A gasoline station had a 1000-gallon tank containing $949\frac{7}{8}$ gallons of gasoline. The four following days the owner sold, respectively, $175\frac{2}{3}$ gallons, $215\frac{5}{8}$ gallons, $167\frac{7}{8}$ gallons, and $223\frac{1}{5}$ gallons. The next day he filled the tank. How much did he put into the tank?

Solution

Instruction	*Operation*
Step 1 Add the four days' sales; whole numbers first.	**Step 1** $175+215+167+223=780$
Step 2 Add the fractions of the four days' sales. Find L.C.D. of fractions using **Rule 9.**	**Step 2** $\begin{array}{r\|llll} 2 & 3 & 8 & 8 & 5 \\ \hline 2 & 3 & 4 & 4 & 5 \\ \hline 2 & 3 & 2 & 2 & 5 \\ \hline & 3 & 1 & 1 & 5 \end{array}$ $2\times2\times2\times3\times1\times1\times5=120$, L.C.D.
Step 3 Reduce the fractions to L.C.D., following **Rule 10.**	**Step 3** $120\div3=40$ and $\frac{2}{3}=\frac{2\times40}{3\times40}=\frac{80}{120}$ $120\div8=15$ and $\frac{5}{8}=\frac{5\times15}{8\times15}=\frac{75}{120}$ $120\div8=15$ and $\frac{7}{8}=\frac{7\times15}{8\times15}=\frac{105}{120}$ $120\div5=24$ and $\frac{1}{5}=\frac{1\times24}{5\times24}=\frac{24}{120}$
Step 4 Add the fractions and reduce to mixed number. Use **Rules 11** and **6.**	**Step 4** $\frac{80}{120}+\frac{75}{120}+\frac{105}{120}+\frac{24}{120}=\frac{284}{120}$ $\frac{284}{120}=2\frac{44}{120}=2\frac{11}{30}$

Step 5

Add the sum of the whole numbers of the four days' sales (**Step** 1) to the result of **Step 4**. (Lesson 7).

Step 5

$$\begin{array}{r} 780 \\ 2\frac{11}{30} \\ \hline 782\frac{11}{30} \end{array}$$

Step 6

Subtract the fraction of **Step 5** from the fraction in the tank contents.

Step 6

$$\frac{7}{8}-\frac{11}{30}$$

Use same L.C.D. as before, or 120

Step 7

Reduce fractions to L.C.D.

Step 7

$$120 \div 8 = 15 \text{ and } \frac{7}{8}=\frac{7\times 15}{8\times 15}=\frac{105}{120}$$

$$120 \div 30 = 4 \text{ and } \frac{11}{30}=\frac{11\times 4}{30\times 4}=\frac{44}{120}$$

Step 8

Subtract fractions.

Step 8

$$\frac{105}{120}-\frac{44}{120}=\frac{61}{120}$$

Step 9

Subtract the whole number of **Step 5** from the whole number of tank contents, then add on the fraction of **Step 8.**

Step 9

$$949-782=167$$

$$167+\frac{61}{120}=167\frac{61}{120}$$

Then $167\frac{61}{120}$ is the number of gallons of gasoline left in the tank before refilling.

Step 10

Subtract the result of **Step 9** from the tank capacity of 1000 gallons. First, take one unit from 1000 (leaving 999) and subtract the fraction from 1.

Step 10

$$1=\frac{120}{120}$$

$$\frac{120}{120}-\frac{61}{120}=\frac{59}{120}$$

Step 11

Subtract the whole numbers and add on the fractional result.

Step 11

$$999-167=832$$

$$832+\frac{59}{120}=832\frac{59}{120} \text{ gals. Ans.}$$

Note: If you have studied the last four problems carefully step by step, you should be able to add and subtract any numbers.

Be sure actually to work all Illustrative Examples. Be sure you can work all of them without looking at the solutions. This gives you practice and experience.

PRACTICE PROBLEMS

After you have worked the following problems compare your answers with the correct answers shown on Page 47.

1. Subtract $2\frac{1}{2}$ from $5\frac{3}{4}$.
2. Subtract $8\frac{2}{9}$ from $14\frac{5}{18}$.
3. Subtract $3\frac{3}{5}$ from $6\frac{7}{8}$.
4. Subtract $1\frac{3}{10}$ from $4\frac{2}{3}$.
5. Subtract $7\frac{5}{16}$ from $12\frac{7}{8}$.
6. Subtract $28\frac{1}{3}$ from $47\frac{2}{5}$.
7. Subtract $10\frac{5}{12}$ from $14\frac{3}{4}$.
8. A farmer, having $450\frac{7}{10}$ acres of land, sold $304\frac{3}{4}$ acres. How many acres did he have left?
9. From the sum $\frac{5}{7}$ and $3\frac{1}{2}$, subtract the difference between $4\frac{1}{3}$ and $5\frac{1}{4}$.
10. A gasoline tank contained $31\frac{1}{2}$ gallons. If $17\frac{5}{8}$ gallons were used how many gallons were left?

ANSWERS TO PRACTICE PROBLEMS

Lesson 1, Page 11

1. $\frac{50}{60}$. 2. $\frac{60}{64}$. 3. $\frac{32}{40}$. 4. $\frac{36}{100}$. 5. $\frac{28}{48}$. 6. $\frac{45}{80}$. 7. $\frac{84}{90}$. 8. $\frac{49}{56}$. 9. $\frac{21}{77}$. 10. $\frac{60}{72}$.

Lesson 2, Page 13

1. $\frac{3}{4}$. 2. $\frac{9}{10}$. 3. $\frac{4}{6}$. 4. $\frac{2}{3}$. 5. $\frac{3}{9}$. 6. $\frac{4}{5}$. 7. $\frac{5}{11}$. 8. $\frac{10}{15}$. 9. $\frac{4}{19}$. 10. $\frac{5}{23}$.

Lesson 3, Page 16

1. $\frac{11}{15}$. 2. $\frac{12}{13}$. 3. $\frac{25}{64}$. 4. $\frac{1}{5}$. 5. $\frac{5}{11}$. 6. $\frac{1}{2}$. 7. $\frac{1}{8}$. 8. $\frac{17}{23}$. 9. $\frac{13}{17}$. 10. $\frac{1}{4}$.

Lesson 4, Page 21

1. $\frac{59}{7}$. 2. $\frac{113}{9}$. 3. $\frac{239}{11}$. 4. $\frac{6005}{6}$. 5. $\frac{250}{3}$. 6. $\frac{384}{5}$. 7. $\frac{467}{10}$. 8. $\frac{797}{14}$. 9. $\frac{265}{7}$. 10. $\frac{1823}{76}$. 11. $1\frac{3}{5}$. 12. $62\frac{1}{2}$. 13. $7\frac{7}{10}$. 14. $18\frac{3}{11}$. 15. $6\frac{6}{7}$. 16. $42\frac{7}{18}$. 17. $66\frac{2}{3}$. 18. $14\frac{1}{2}$. 19. $13\frac{3}{7}$. 20. $21\frac{1}{2}$.

Lesson 5, Page 26

1. 24. 2. 48. 3. 60. 4. 1,001. 5. 2,805. 6. 240. 7. 27,720. 8. 300. 9. 60. 10. 24.

Lesson 5, Page 29

1. $\frac{24}{168}, \frac{63}{168}, \frac{42}{168}, \frac{112}{168}$. 2. $\frac{24}{84}, \frac{28}{84}, \frac{63}{84}, \frac{42}{84}$. 3. $\frac{24}{72}, \frac{16}{72}, \frac{30}{72}, \frac{27}{72}$. 4. $\frac{10}{150}, \frac{45}{150}, \frac{24}{150}, \frac{5}{150}$. 5. $\frac{77}{231}, \frac{66}{231}, \frac{84}{231}$.

Lesson 6, Page 34

1. $\frac{37}{60}$. 2. $2\frac{1}{40}$. 3. $2\frac{17}{18}$. 4. $2\frac{7}{12}$. 5. $2\frac{3}{4}$. 6. $1\frac{34}{45}$. 7. $1\frac{61}{120}$. 8. $1\frac{13}{28}$. 9. $1\frac{23}{40}$. 10. $1\frac{53}{60}$. 11. $1\frac{43}{63}$. 12. $1\frac{6}{7}$.

Lesson 7, Page 38

1. $11\frac{2}{3}$. 2. $7\frac{1}{15}$. 3. $13\frac{23}{105}$. 4. $6\frac{3}{8}$. 5. $12\frac{1}{2}$. 6. $30\frac{7}{24}$. 7. $14\frac{79}{280}$. 8. $14\frac{31}{84}$. 9. $19\frac{49}{90}$. 10. $26\frac{11}{72}$. 11. $5\frac{25}{96}$. 12. $23\frac{11}{12}$.

Lesson 8, Page 41

1. $\frac{1}{3}$. 2. $\frac{1}{4}$. 3. $\frac{14}{27}$. 4. $\frac{13}{70}$. 5. $\frac{5}{18}$. 6. $\frac{19}{136}$. 7. $\frac{41}{70}$. 8. $\frac{2}{35}$. 9. $\frac{5}{12}$. 10. $\frac{1}{7}$.

Lesson 9, Page 46

1. $3\frac{1}{4}$. 2. $6\frac{1}{18}$. 3. $3\frac{11}{40}$. 4. $3\frac{11}{30}$. 5. $5\frac{9}{16}$. 6. $19\frac{1}{15}$. 7. $4\frac{1}{3}$. 8. $145\frac{19}{20}$. 9. $3\frac{25}{84}$. 10. $13\frac{7}{8}$.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on Page 50.

If you miss more than two of the problems it means that you should review the whole book very carefully.

Do not try this trial examination until you have worked every Practice Problem in this Section.

Do not start the final examination until you have completed this trial examination.

1. Change the following fractions to higher terms using 2 as a multiplier.

$$\frac{3}{10},\ \frac{6}{7},\ \frac{8}{23},\ \frac{10}{50},\ \frac{14}{33}$$

2. Find the L.C.D. for the following fractions.

$$\frac{3}{4},\ \frac{3}{8},\ \frac{1}{2},\ \frac{5}{16},\ \frac{1}{4}$$

3. Reduce the following fractions to L.C.D.

$$\frac{5}{8},\ \frac{7}{32},\ \frac{13}{64},\ \frac{1}{16},\ \frac{3}{4}$$

4. Add the following fractions.

$$\frac{5}{6}+\frac{3}{8}+\frac{2}{3}+\frac{1}{2}+\frac{1}{5}+\frac{2}{9}$$

5. Use cancellation to reduce the following fractions to their lowest terms.

$$\frac{10}{75},\ \frac{100}{120},\ \frac{56}{78},\ \frac{25}{125},\ \frac{16}{40}$$

6. Change the following fractions to mixed numbers.

$$\frac{102}{8},\ \frac{125}{55},\ \frac{15}{4},\ \frac{78}{5},\ \frac{46}{3}$$

7. Change the following mixed numbers to improper fractions.

$$7\tfrac{1}{10},\ 16\tfrac{2}{3},\ 50\tfrac{1}{4},\ 28\tfrac{7}{8},\ 5\tfrac{13}{15}$$

8. Subtract $43\frac{3}{4}$ from $68\frac{3}{8}$.

9. An electrical contractor bought a reel of 1000 feet of wire. On one job he used $125\frac{1}{2}$ feet and $118\frac{3}{4}$ feet. On another job he used $174\frac{2}{3}$ feet and $236\frac{5}{6}$ feet. How many feet of wire did he have left on the reel?

10. In an athletic contest six men won the following points: $10\frac{4}{7}$, $9\frac{3}{16}$, $8\frac{2}{9}$, $7\frac{3}{8}$, $7\frac{1}{4}$, and $6\frac{2}{3}$. How many points were left for all others, out of a possible 60 points?

FINAL EXAMINATION

1. Change the following fractions to higher terms using 3 as a multiplier.

$$\frac{10}{15},\ \frac{14}{72},\ \frac{72}{93}$$

2. Find the L.C.D. for the following fractions.

$$\frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \frac{1}{9}$$

3. Reduce the following fractions to L.C.D.

$$\frac{7}{8},\ \frac{3}{4},\ \frac{5}{12},\ \frac{5}{6}$$

4. Reduce the following fractions to L.C.D.

$$\frac{2}{3},\ \frac{7}{10},\ \frac{4}{5},\ \frac{4}{15}$$

5. Use cancellation method to reduce the following fractions to their lowest terms.

$$\frac{100}{150},\ \frac{128}{160},\ \frac{156}{180}$$

6. Change the following fractions to mixed numbers.

$$\frac{101}{8},\ \frac{125}{6},\ \frac{184}{7},\ \frac{124}{5}$$

7. Change the following mixed numbers to improper fractions.

$$6\tfrac{3}{4},\ 5\tfrac{5}{12},\ 8\tfrac{8}{9},\ 10\tfrac{11}{16}$$

8. (a) Subtract $7\frac{7}{11}$ from $25\frac{12}{13}$. (b) Subtract $5\frac{6}{7}$ from $15\frac{4}{21}$.

9. (a) Add $\frac{3}{4}+\frac{3}{8}+\frac{5}{16}+\frac{7}{32}$. (b) Add $\frac{3}{20}+\frac{4}{5}+\frac{7}{30}+\frac{1}{6}$.

10. A machinist had to cut five pieces from a bar of steel 25 feet long. How much would be left if the pieces were of the following lengths:

$3\frac{1}{2}$ feet, $3\frac{7}{12}$ feet, $4\frac{1}{6}$ feet, $4\frac{1}{4}$ feet, and $4\frac{2}{3}$ feet.

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. Review **Rule 2.** Following this rule, and using 2 as a multiplier, the solution is as follows:

$$\frac{3}{10}=\frac{3\times2}{10\times2}=\frac{6}{20}$$

$$\frac{6}{7}=\frac{6\times2}{7\times2}=\frac{12}{14}$$

$$\frac{8}{23}=\frac{8\times2}{23\times2}=\frac{16}{46}$$

$$\frac{10}{50}=\frac{10\times2}{50\times2}=\frac{20}{100}$$

$$\frac{14}{33}=\frac{14\times2}{33\times2}=\frac{28}{66}$$

2. Review **Rule 9.**

$$\begin{array}{r|rrrrr}
2) & 4 & 8 & 2 & 16 & 4 \\ \hline
2) & 2 & 4 & 1 & 8 & 2 \\ \hline
2) & 1 & 2 & 1 & 4 & 1 \\ \hline
 & 1 & 1 & 1 & 2 & 1
\end{array}$$

$2\times2\times2\times1\times1\times1\times2\times1=16$, L.C.D.

3. To solve this problem you must use **Rules 9** and **10.**
First find L.C.D. using **Rule 9.**

$$\begin{array}{r|rrrrr}
2) & 8 & 32 & 64 & 16 & 4 \\ \hline
2) & 4 & 16 & 32 & 8 & 2 \\ \hline
2) & 2 & 8 & 16 & 4 & 1 \\ \hline
2) & 1 & 4 & 8 & 2 & 1 \\ \hline
2) & 1 & 2 & 4 & 1 & 1 \\ \hline
 & 1 & 1 & 2 & 1 & 1
\end{array}$$

$2\times2\times2\times2\times2\times1\times1\times2\times1\times1=64$, L.C.D.

Next reduce fractions to L.C.D. using **Rule 10.**

$64\div 8= 8$ and $\frac{5}{8}=\frac{5\times8}{8\times8}=\frac{40}{64}$
$64\div32= 2$ and $\frac{7}{32}=\frac{7\times2}{32\times2}=\frac{14}{64}$
$64\div64= 1$ and $\frac{13}{64}=\frac{13\times1}{64\times1}=\frac{13}{64}$
$64\div16= 4$ and $\frac{1}{16}=\frac{1\times4}{16\times4}=\frac{4}{64}$
$64\div 4=16$ and $\frac{3}{4}=\frac{3\times16}{4\times16}=\frac{48}{64}$

4. Use **Rules 9, 10,** and **11.**
First use **Rule 9.**

$$\begin{array}{r|rrrrrr}
2) & 6 & 8 & 3 & 2 & 5 & 9 \\ \hline
3) & 3 & 4 & 3 & 1 & 5 & 9 \\ \hline
 & 1 & 4 & 1 & 1 & 5 & 3
\end{array}$$

$2\times3\times1\times4\times1\times1\times5\times3=360$, L.C.D.

Next use **Rule 10.**

$$360 \div 6 = 60 \text{ and } \tfrac{5}{6} = \tfrac{5 \times 60}{6 \times 60} = \tfrac{300}{360}$$
$$360 \div 8 = 45 \text{ and } \tfrac{3}{8} = \tfrac{3 \times 45}{8 \times 45} = \tfrac{135}{360}$$
$$360 \div 3 = 120 \text{ and } \tfrac{2}{3} = \tfrac{2 \times 120}{3 \times 120} = \tfrac{240}{360}$$
$$360 \div 2 = 180 \text{ and } \tfrac{1}{2} = \tfrac{1 \times 180}{2 \times 180} = \tfrac{180}{360}$$
$$360 \div 5 = 72 \text{ and } \tfrac{1}{5} = \tfrac{1 \times 72}{5 \times 72} = \tfrac{72}{360}$$
$$360 \div 9 = 40 \text{ and } \tfrac{2}{9} = \tfrac{2 \times 40}{9 \times 40} = \tfrac{80}{360}$$

Next add the fractions as explained in **Rule 11** and Lesson 6.

$$\tfrac{300}{360} + \tfrac{135}{360} + \tfrac{240}{360} + \tfrac{180}{360} + \tfrac{72}{360} + \tfrac{80}{360} = 300 + 135 + 240 + 180 + 72 + 80 = \tfrac{1007}{360}$$

Then $\frac{1007}{360} = 2\frac{287}{360}$ (**Rule** 8) Ans.

5. Use **Rule 6.**

$$\begin{array}{c} 2 \\ \cancel{10} \\ \hline \cancel{75} \\ 15 \end{array} = \tfrac{2}{15} \qquad \begin{array}{c} 5 \\ \cancel{25} \\ \cancel{50} \\ \cancel{100} \\ \hline \cancel{120} \\ \cancel{60} \\ \cancel{30} \\ 6 \end{array} = \tfrac{5}{6} \qquad \begin{array}{c} 28 \\ \cancel{56} \\ \hline \cancel{78} \\ 39 \end{array} = \tfrac{28}{39} \qquad \begin{array}{c} 1 \\ \cancel{5} \\ \cancel{25} \\ \hline \cancel{125} \\ \cancel{25} \\ 5 \end{array} = \tfrac{1}{5} \qquad \begin{array}{c} 2 \\ \cancel{4} \\ \cancel{8} \\ \cancel{16} \\ \hline \cancel{40} \\ \cancel{20} \\ \cancel{10} \\ 5 \end{array} = \tfrac{2}{5}$$

6. Use **Rule 8.**

$\frac{102}{8} = 102 \div 8 = 12\frac{3}{4}$ Ans. **(Rule 6)**
$\frac{125}{55} = 125 \div 55 = 2\frac{3}{11}$ Ans.
$\frac{15}{4} = 15 \div 4 = 3\frac{3}{4}$ Ans.
$\frac{78}{5} = 78 \div 5 = 15\frac{3}{5}$ Ans.
$\frac{46}{3} = 46 \div 3 = 15\frac{1}{3}$ Ans.

7. Use **Rule 7.**

$$7\tfrac{1}{10} = \tfrac{7 \times 10 + 1}{10} = \tfrac{71}{10}$$
$$16\tfrac{2}{3} = \tfrac{16 \times 3 + 2}{3} = \tfrac{50}{3}$$
$$50\tfrac{1}{4} = \tfrac{50 \times 4 + 1}{4} = \tfrac{201}{4}$$
$$28\tfrac{7}{8} = \tfrac{28 \times 8 + 7}{8} = \tfrac{231}{8}$$
$$5\tfrac{13}{15} = \tfrac{5 \times 15 + 13}{15} = \tfrac{88}{15}$$

8. Follow the procedure given for Illustrative Example 2 of Lesson 9, using **Rules 9, 10, 13, and 14.**

By **Rule 9.**

$$\text{L.C.D.} = 8$$

By **Rule 10.**

$$8 \div 4 = 2 \text{ and } \tfrac{3}{4} = \tfrac{3 \times 2}{4 \times 2} = \tfrac{6}{8}$$
$$8 \div 8 = 1 \text{ and } \tfrac{3}{8} = \tfrac{3 \times 1}{8 \times 1} = \tfrac{3}{8}$$

In this problem $68\frac{3}{8}$ is the minuend and $43\frac{3}{4}$ is the subtrahend. After changing the fractions to L.C.D. the problem can be stated $68\frac{3}{8} - 43\frac{6}{8}$.

To follow **Rule 14** we must first subtract the fractions. However we cannot subtract $\frac{6}{8}$ from $\frac{3}{8}$. Therefore we have to take 1 from the 68, leaving it 67, and add the 1 to $\frac{3}{8}$ (as explained in Lesson 9) and make $1\frac{3}{8}$. Then by **Rule 7**

$$1\tfrac{3}{8}=\frac{1\times 8+3}{8}=\frac{11}{8}$$

Now we can subtract $\frac{6}{8}$ from $\frac{11}{8}$ and have $\frac{5}{8}$.

Next subtract whole numbers

$$67-43=24$$

Then add the remainders from the fractions and whole numbers.

$$24+\tfrac{5}{8}=24\tfrac{5}{8}\text{ Ans.}$$

9. After reading and thinking of this problem we can reason out that several pieces or lengths of wire were cut off the reel and that we can find the sum of all these pieces and subtract this sum from 1000 to find out how much was left on the reel. The first step in the solution is to add all of the pieces which were cut off. These are as follows.

$$125\tfrac{1}{2}+118\tfrac{3}{4}+174\tfrac{2}{3}+236\tfrac{5}{6}$$

Lesson 7 explains how to add mixed numbers.

First add the whole numbers

$$125+118+174+236=653$$

Then add the fractions

$$\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{2}{3}+\tfrac{5}{6}$$

Use **Rule 9.**

$$\text{L.C.D.}=12$$

Use **Rule 10.**

$$12\div 2=6 \text{ and } \tfrac{1}{2}=\tfrac{1\times 6}{2\times 6}=\tfrac{6}{12}$$
$$12\div 4=3 \text{ and } \tfrac{3}{4}=\tfrac{3\times 3}{4\times 3}=\tfrac{9}{12}$$
$$12\div 3=4 \text{ and } \tfrac{2}{3}=\tfrac{2\times 4}{3\times 4}=\tfrac{8}{12}$$
$$12\div 6=2 \text{ and } \tfrac{5}{6}=\tfrac{5\times 2}{6\times 2}=\tfrac{10}{12}$$

Use **Rule 11.**

$$\tfrac{6}{12}+\tfrac{9}{12}+\tfrac{8}{12}+\tfrac{10}{12}=\tfrac{33}{12}$$

By **Rule 8** we know that $\frac{33}{12}=2\frac{9}{12}$

By **Rule 6** we know that $2\frac{9}{12}=2\frac{3}{4}$

Next we add the sum of the whole numbers to the sum of the fractions.

$$653+2\tfrac{3}{4}=655\tfrac{3}{4}$$

The final step of the problem is to subtract $655\frac{3}{4}$ from 1000. Use **Rule 14.** This step can be written

$$1000-655\tfrac{3}{4}$$

The 1000 is the minuend and the $655\frac{3}{4}$ is the subtrahend. There is no fraction in the minuend so we take 1 from the 1000, leaving it 999. In terms of 4ths this 1 is $\frac{4}{4}$. Then the problem is

$$999\tfrac{4}{4}-655\tfrac{3}{4}$$

Subtracting the fractions

$$\tfrac{4}{4}-\tfrac{3}{4}=\tfrac{1}{4}$$

Subtracting whole numbers

$$999 - 655 = 344$$

Adding remainders of fractions and whole numbers

$$344 + \tfrac{1}{4} = 344\tfrac{1}{4} \text{ feet. Ans.}$$

10. This problem requires that we subtract the sum of all the mixed numbers from 60. So we must first add the mixed numbers.

Use **Rules 9** and **10** and method of Lesson 7.

Adding whole numbers

$$10+9+8+7+7+6=47$$

Adding the fractions

Use **Rule 9.**

$$\begin{array}{r|llllll} 2) & 7 & 16 & 9 & 8 & 4 & 3 \\ \hline 2) & 7 & 8 & 9 & 4 & 2 & 3 \\ \hline 2) & 7 & 4 & 9 & 2 & 1 & 3 \\ \hline 3) & 7 & 2 & 9 & 1 & 1 & 3 \\ \hline & 7 & 2 & 3 & 1 & 1 & 1 \end{array}$$

$$2\times 2\times 2\times 3\times 7\times 2\times 3\times 1\times 1\times 1=1008, \text{ L.C.D.}$$

Use **Rule 10.**

$$1008 \div 7 = 144 \text{ and } \tfrac{4}{7} = \tfrac{4\times 144}{7\times 144} = \tfrac{576}{1008}$$
$$1008 \div 16 = 63 \text{ and } \tfrac{3}{16} = \tfrac{3\times 63}{16\times 63} = \tfrac{189}{1008}$$
$$1008 \div 9 = 112 \text{ and } \tfrac{2}{9} = \tfrac{2\times 112}{9\times 112} = \tfrac{224}{1008}$$
$$1008 \div 8 = 126 \text{ and } \tfrac{3}{8} = \tfrac{3\times 126}{8\times 126} = \tfrac{378}{1008}$$
$$1008 \div 4 = 252 \text{ and } \tfrac{1}{4} = \tfrac{1\times 252}{4\times 252} = \tfrac{252}{1008}$$
$$1008 \div 3 = 336 \text{ and } \tfrac{2}{3} = \tfrac{2\times 336}{3\times 336} = \tfrac{672}{1008}$$

$$\tfrac{576}{1008}+\tfrac{189}{1008}+\tfrac{224}{1008}+\tfrac{378}{1008}+\tfrac{252}{1008}+\tfrac{672}{1008}=$$
$$576+189+224+378+252+672=\tfrac{2291}{1008}$$

By **Rule 8.**

$$\tfrac{2291}{1008} = 2\tfrac{275}{1008}$$

Then add the sum of the whole numbers and fractions.

$$47+2\tfrac{275}{1008}=49\tfrac{275}{1008}$$

This $49\frac{275}{1008}$ is the sum of all the points won by the six men.

Finally we must subtract $49\frac{275}{1008}$ from 60. The minuend (60) has no fraction, so we must take 1 from 60, leaving it 59. This 1, expressed in terms of 1008's equals $\frac{1008}{1008}$. The problem can now be written

$$59\tfrac{1008}{1008}-49\tfrac{275}{1008}=10\tfrac{733}{1008} \text{ Ans.}$$

PRACTICAL MATHEMATICS

Section 4

FRACTIONS—Part II

Lesson 1

For Step 1, recall multiplication of numbers, cancellation, and what a fraction is. For Step 2, learn how to multiply a fraction by a whole number. For Step 3, work Practice Problems 1 to 4. For Step 4, work Practice Problems 1 to 18.

MULTIPLICATION OF FRACTIONS

When you studied multiplication, in Section 1 (Book 1), you learned that a **product** is the result obtained when two numbers are multiplied together. Thus (*a*) $2 \times 2 = 4$, (*b*) $2 \times 18 = 36$, (*c*) $9 \times 9 = 81$. The product, in example (*a*) is 4, in example (*b*) is 36, in example (*c*) is 81.

When you studied Section 3 you learned how to add and subtract fractions. In this section you will learn how to do the multiplication that is often required where fractions are concerned and also how to divide fractions.

When we multiply one fraction by another fraction or a fraction by a whole or mixed number, etc., the answer is called the **product**, the same as when whole numbers are multiplied. Therefore always keep in mind that the product is the result of multiplication, whether for fractions or for whole or mixed numbers.

Up to this point you have learned that the word "times" (meaning *to multiply*) is represented by the symbol $\times$. Thus if we want to multiply 2 by 2 we can write it *2 times 2* or 2×2. The 2×2 is used because $\times$ is a standard symbol for multiplication.

In working with fractions we also use the symbol $\times$ to denote that a fraction is to be multiplied by some other fraction or number. Thus $\frac{1}{2} \times \frac{1}{3}$ or $\frac{1}{10} \times 4$, etc., indicates multiplication. Also you will find that sometimes we have an expression like, $\frac{1}{2}$ of $\frac{1}{3}$. *Remember that in working with fractions the word "of" means the same as "times" or $\times$ and that all three mean multiplication.*

There are several combinations possible where multiplication and fractions are concerned. Such combinations are as follows:

$\frac{1}{4} \times 10$ (Fraction and a whole number)
$\frac{1}{4} \times \frac{1}{8}$ (Two fractions)
$6\frac{1}{3} \times 16$ (A mixed number and a whole number)
$10\frac{1}{5} \times 4\frac{1}{2}$ (Two mixed numbers)

There is a different rule for each of these four combinations and all are explained in the following.

Multiplying a Fraction and a Whole Number. You have already learned that the denominator (lower part) of a fraction indicates into how many equal parts a unit has been divided, and that the

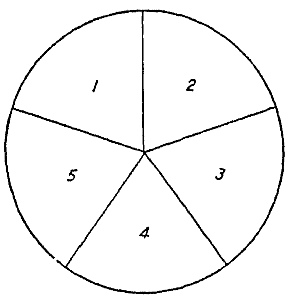


Fig. 1

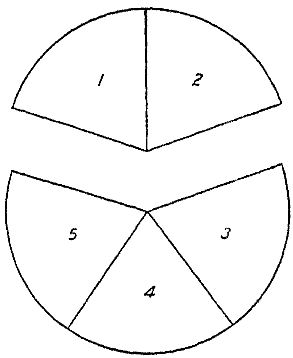


Fig. 2

numerator (upper part) indicates how many of those parts we are thinking of. As an example of this, note Fig. 1. Imagine that this figure represents a pie which has been cut so as to make 5 pieces as shown. This illustrates something being divided into equal parts. In this particular case there are 5 equal parts. Now note Fig. 2. Here we see that 2 of the pieces of pie have been removed. Or, $\frac{2}{5}$ of the pie has been removed. This illustrates the number of parts we are thinking of. In other words in the fraction $\frac{2}{5}$, when thought of in terms of the pie, the denominator (5) represents the number of pieces into which the pie is divided, and the numerator (2) represents the number of those equal pieces we took away.

This same explanation would apply if any object other than a pie were considered or other fractions used.

Now that we understand what the numerator and denominator

of a fraction represent, we can proceed to find out how to multiply a fraction by a whole number.

Suppose we wish to multiply $\frac{2}{5}$ by 2. This is written $\frac{2}{5}\times 2$. You should learn at this point that when we multiply a fraction by a whole number we multiply only the numerator of the fraction by the whole number. This is done as follows:

$$\frac{2}{5}\times 2=\frac{2\times 2}{5}=\frac{4}{5}$$

In doing the above operation all we did was to multiply the numerator of the fraction, which is 2, by the multiplier, which is 2. Now we can state a rule which we can follow whenever we want to multiply a fraction by a whole number.

Rule 1. *To multiply any fraction and any whole number, multiply the numerator of the fraction by the whole number.*

The product thus obtained is the numerator of the answer. The denominator of the answer is the same as that of the fraction before multiplication.

ILLUSTRATIVE EXAMPLES

1. Multiply $\frac{1}{8}$ by 3.

The various parts of this example are as follows:

$$\begin{matrix}\text{(numerator)} \\ \text{(denominator)}\end{matrix}\ \frac{1}{8}\times 3\ \text{(whole number)}$$

The solution, following **Rule 1,** is

$$\frac{1}{8}\times 3=\frac{1\times 3}{8}=\frac{3}{8}$$

2. Multiply $\frac{3}{7}$ by 2.

The solution, following **Rule 1,** is

$$\frac{3}{7}\times 2=\frac{3\times 2}{7}=\frac{6}{7}$$

In the above examples we have followed **Rule 1.** Remember that "by" means the same as $\times$. In each case we multiplied the numerator of the fraction by the whole number. The product of the multiplication became the numerator for the final answers. The denominator remained the same as in the fraction before multiplication.

PRACTICE PROBLEMS

1. Multiply $\frac{2}{9}$ by 2. Ans. $\frac{4}{9}$

2. Multiply $\frac{1}{10}$ by 3. Ans. $\frac{3}{10}$

3. Multiply $\frac{1}{20}$ by 7. Ans. $\frac{7}{20}$

4. Multiply $\frac{3}{11}$ by 3. Ans. $\frac{9}{11}$

So far we have worked with fractions of a nature such that the answer was always in "lowest terms." (If this expression is not clear to you refer to Section 3 and review the part which presents the method of reducing fractions to lowest terms.) Sometimes when we multiply a fraction by a whole number we get an answer that is not in its lowest terms. When this happens we should always reduce the answer to lowest terms.

ILLUSTRATIVE EXAMPLES

1. Multiply $\frac{3}{20}$ by 6. Following **Rule 1,**

$$\frac{3}{20}\times 6=\frac{3\times 6}{20}=\frac{18}{20}$$

Here the $\frac{18}{20}$ can be reduced to lowest terms. This reduction is done by cancellation as explained in Section 2.

Thus

$$\frac{\overset{9}{\cancel{18}}}{\underset{10}{\cancel{20}}}=\frac{9}{10}$$

Here we found, by trial, that the 18 and the 20 could both be divided exactly by 2. The final answer is $\frac{9}{10}$ because no number will divide both 9 and 10 an exact number of times.

2. Multiply $\frac{5}{10}$ by 10.

$$\frac{5}{10}\times 10=\frac{5\times 10}{10}=\frac{50}{10}=5$$

Here we have a slightly different situation in that $\frac{50}{10}$ is an *improper* fraction. (Review Section 3 if this term is not clear to you.) Therefore we must divide the numerator (50) by the denominator (10).

$$\frac{50}{10}=50\div 10=5$$

Cancellation Method. There is an easier and quicker method of multiplying a fraction by a whole number that can be used in many cases. Take the last illustrative example we worked, namely, $\frac{5}{10}\times 10$. By applying the rules of cancellation we can greatly shorten this multiplication.

$$\frac{5}{10}\times 10=\frac{5}{\underset{1}{\cancel{10}}}\times \overset{1}{\cancel{10}}=\frac{5}{1}\times 1=\frac{5\times 1}{1}=\frac{5}{1}=5$$

Here we cancelled the two tens because both of them could be divided by 10. After cancelling we had $\frac{5}{1}\times 1$ left and we see at once, by following **Rule 1**, that the answer is $\frac{5}{1}$ or 5. This is true because we learned in Section 3 that $\frac{5}{1}$ means $5\div 1$ and $5\div 1=5$.

3. Multiply $\frac{11}{72}$ by 18.

$$\frac{11}{\underset{\underset{4}{\cancel{8}}}{\cancel{72}}}\times \overset{\overset{1}{\cancel{2}}}{\cancel{18}}=\frac{11}{4}\times 1=\frac{11\times 1}{4}=\frac{11}{4}=2\tfrac{3}{4}$$

By trial we found that 9 would divide an exact number of times into both 18 and 72. This left 2 and 8. Again by trial we found that 2 would divide an exact number of times into 2 and 8 leaving 1 and 4. Then we followed **Rule 1.** (You learned in Sections 2 and 3 that there are sometimes several ways of cancelling by division. For example, we could have divided both 18 and 72 by 18.) We thus have $\frac{11}{4}$. However, this is an improper fraction and should be changed to a mixed number. Thus $11\div 4=2\frac{3}{4}$.

In the following, although the **Rule 1** step is not shown, it has been followed.

4. Multiply $\frac{5}{32}$ by 16.

$$\frac{5}{\underset{2}{\cancel{32}}}\times \overset{1}{\cancel{16}}=\frac{5}{2}=2\tfrac{1}{2}$$

Here we divided 16 and 32 by 16. As explained above, there are other ways of cancelling. We might have done it like this:

$$\frac{5}{\underset{\underset{2}{\cancel{4}}}{\cancel{32}}}\times \overset{\overset{1}{\cancel{2}}}{\cancel{16}}=\frac{5}{2}=2\tfrac{1}{2}$$

Here we divided the 16 and 32 by 8. This left 2 and 4 which we divided by 2. We could also have done it like this:

$$\frac{5}{\underset{\underset{2}{\cancel{8}}}{\cancel{32}}}\times\overset{\overset{1}{\cancel{4}}}{\cancel{16}}=\frac{5}{2}=2\tfrac{1}{2}$$

Here we divided 16 and 32 by 4. This left 4 and 8 which we divided by 4.

Note that although three ways of cancelling are shown for this example, all give the same answer.

5. Multiply $\frac{10}{32}$ by 20.

$$\frac{\overset{5}{\cancel{10}}}{\underset{\underset{4}{\cancel{8}}}{\cancel{32}}}\times\overset{5}{\cancel{20}}=\frac{25}{4}=6\tfrac{1}{4}$$

Here we divided the 20 and 32 by 4. Then we divided 10 and 8 by 2. We could also have done it like this

$$\frac{\overset{5}{\cancel{10}}}{\underset{\underset{\underset{4}{\cancel{8}}}{\cancel{16}}}{\cancel{32}}}\times\overset{\overset{5}{\cancel{10}}}{\cancel{20}}=\frac{25}{4}=6\tfrac{1}{4}$$

Here we divided the 20 and 32 by 2 and had 10 and 16 left. We then divided the 10 and 16 by 2 and had 5 and 8 left. Then we divided 10 and 8 by 2.

For beginners in mathematics it is sometimes easier to use the smallest possible number for dividing. This saves time in finding numbers that will divide both the required numbers an exact number of times. As you get more experience you will gradually learn to use larger numbers. However, you can use either large or small numbers and the final results will be the same, as illustrated in foregoing Examples 4 and 5.

There is one more condition in the multiplication of fractions by whole numbers to understand. Multiply $\frac{1}{8}$ by 2.

$$\frac{1}{8}\times 2=\frac{1\times 2}{8}=\frac{2}{8}=\frac{1}{4}\text{ or }\frac{1}{\underset{4}{\cancel{8}}}\times\overset{1}{\cancel{2}}=\frac{1}{4}$$

Here we multiplied $\frac{1}{8}$ by 2 and got an answer of $\frac{1}{4}$. We may wonder why our answer is in terms of a smaller denominator than that with which we began. In other words we might wonder why we begin with a denominator of 8 and, after multiplying, have a denominator of only 4. To understand this requires that we make an experiment.

Note Fig. 3. To make this figure, assume that first we drew a line from A to B. The space above the line is divided into 4 equal parts. Thus each part is exactly $\frac{1}{4}$ of the line AB. The space below AB is divided into 8 equal parts, each part is exactly $\frac{1}{8}$ of line AB.

By looking at Fig. 3 we can see that $\frac{1}{4}$ of line AB is twice as long as $\frac{1}{8}$ of line AB. Or, in other words, if we take 2 of the $\frac{1}{8}$*th*

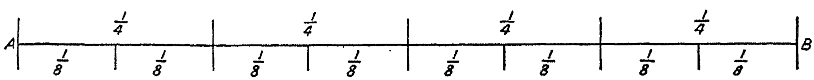


Fig. 3

divisions we see that these two together are only as long as 1 of the $\frac{1}{4}$*th* divisions. Or, if we multiplied 1 of the $\frac{1}{8}$*th* divisions by 2 we would have a length exactly equal to $\frac{1}{4}$ of the line. This shows us why $\frac{1}{8}$ multiplied by 2 equals $\frac{1}{4}$. It also shows that $\frac{1}{4}$ is really larger than $\frac{1}{8}$ even though the denominator of $\frac{1}{4}$ is smaller than the denominator of $\frac{1}{8}$.

This same explanation can be used for other fractions and the student is urged to make drawings of his own if he has difficulty in understanding, for example, why $\frac{1}{10}\times 2=\frac{1}{5}$.

Written Problems. Written problems are the type shown in the following. These problems are no more difficult than are other types and require only a little thought to determine the method of solution.

ILLUSTRATIVE EXAMPLE

1. A farmer had a load of 40 sacks of corn on a truck. If $\frac{1}{5}$ of the sacks fell out of the truck, how many were left?

Solution. In this problem we are to suppose that $\frac{1}{5}$ of 40 sacks fell out of the truck. If we can find $\frac{1}{5}$ of 40 and subtract that amount from 40 we will have the correct answer.

We know that $\frac{1}{5}$ of 40 means the same as $\frac{1}{5}\times 40$, because we learned earlier in this lesson that "of" means the same as $\times$.

We have to multiply a fraction ($\frac{1}{5}$) by a whole number (40).

Following **Rule 1.**

$$\frac{1}{\underset{1}{\cancel{5}}}\times \overset{8}{\cancel{40}}=8$$

The number of sacks that fell out of the truck, then, is 8. To find the number of sacks left in the truck, we subtract 8 from 40, or $40-8=32$.

Another Method: If $\frac{1}{5}$ of the sacks fell out, then $\frac{4}{5}$ of the sacks remained in the truck. Therefore the number remaining is

$$\frac{4}{5}\text{ of }40=\frac{4}{\underset{1}{\cancel{5}}}\times \overset{8}{\cancel{40}}=32$$

PRACTICE PROBLEMS

1. Multiply $\frac{4}{5}$ by 35. Ans. 28
2. Multiply $\frac{5}{84}$ by 84. Ans. 5
3. Multiply $\frac{7}{50}$ by 10. Ans. $1\frac{2}{5}$
4. Multiply $\frac{49}{65}$ by 13. Ans. $9\frac{4}{5}$
5. Multiply $\frac{5}{12}$ by 132. Ans. 55
6. Multiply $\frac{7}{64}$ by 96. Ans. $10\frac{1}{2}$
7. Multiply $\frac{8}{21}$ by 7. Ans. $2\frac{2}{3}$
8. Multiply $\frac{71}{87}$ by 29. Ans. $23\frac{2}{3}$
9. Multiply $\frac{5}{48}$ by 64. Ans. $6\frac{2}{3}$
10. Multiply $\frac{11}{64}$ by 192. Ans. 33
11. Multiply $\frac{23}{100}$ by 25. Ans. $5\frac{3}{4}$
12. Multiply $\frac{151}{450}$ by 15. Ans. $5\frac{1}{30}$
13. Multiply $\frac{7}{21}\times 300$. Ans. 100
14. Multiply $\frac{102}{204}\times 476$. Ans. 238
15. Multiply $\frac{34}{72}\times 384$. Ans. $181\frac{1}{3}$
16. A man owns 176 head of cattle. If he sells $\frac{3}{4}$ of them, how many has he left? Ans. 44 head of cattle.
17. A man starts out to travel 126 miles. After traveling $\frac{2}{3}$ of the distance, how many miles has he to travel? Ans. 42 miles.
18. A man owned one section of land, or 640 acres, and sold $\frac{1}{2}$ of it. How much did he sell? Ans. 320 acres.

Lesson 2

For Step 1, recall what you learned in Lesson 1. For Step 2, learn how to multiply a fraction by a fraction. For Step 3, work Practice Problems 1 to 6. For Step 4, work Practice Problems 1 to 9.

Multiplying a Fraction by a Fraction. In multiplying one fraction by another fraction we use the $\times$ sign as already explained. Thus to multiply $\frac{1}{4}$ by $\frac{1}{2}$ we write it $\frac{1}{4}\times\frac{1}{2}$. We must remember that the $\times$ sign means the same as "of." So, $\frac{1}{4}\times\frac{1}{2}$ means the same as $\frac{1}{4}$ of $\frac{1}{2}$.

Rule 2. *To multiply a fraction by a fraction, multiply the numerator of one fraction by the numerator of the other fraction. Do the same with the denominators.*

ILLUSTRATIVE EXAMPLES

1. Multiply $\frac{1}{2}$ by $\frac{1}{4}$.

Following **Rule 2**

$$\frac{1}{2}\times\frac{1}{4}=\frac{1\times1}{2\times4}=\frac{1}{8}$$

Here we multiplied the numerators ($1\times1=1$) and the denominators ($2\times4=8$) and the answer is $\frac{1}{8}$.

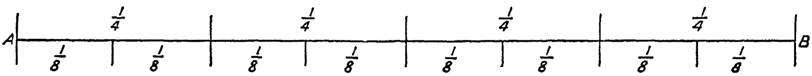


Fig. 4

Now let us make a simple experiment so we can see this process more clearly. Note Fig. 4. The space above the line AB has been divided into 4 equal parts. Each part is $\frac{1}{4}$ of line AB. The space below the line AB has been divided into 8 equal parts. Each part is $\frac{1}{8}$ of line AB.

We know that $\frac{1}{2}\times\frac{1}{4}$ means the same as $\frac{1}{2}$ of $\frac{1}{4}$. Looking at Fig. 4 we can see that if we divide $\frac{1}{4}$ into 2 parts each part is equal to $\frac{1}{8}$. Thus $\frac{1}{2}$ of $\frac{1}{4}=\frac{1}{8}$. Or, $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$ just as we calculated it according to **Rule 2.**

2. Multiply $\frac{1}{2}$ by $\frac{1}{8}$.

$$\frac{1}{2}\times\frac{1}{8}=\frac{1\times1}{2\times8}=\frac{1}{16}$$

This solution follows **Rule 2.** We can illustrate this solution by Fig. 5. We know that $\frac{1}{2}\times\frac{1}{8}$ means the same as $\frac{1}{2}$ of $\frac{1}{8}$. Looking at Fig. 5 we can see that $\frac{1}{2}$ of $\frac{1}{8}$ equals $\frac{1}{16}$.

3. Multiply $\frac{1}{2}$ by $\frac{3}{5}$.

$$\frac{1}{2}\times\frac{3}{5}=\frac{1\times3}{2\times5}=\frac{3}{10}$$

This can be illustrated by Fig. 6. We know that $\frac{1}{2}\times\frac{3}{5}$ is the same as $\frac{1}{2}$ of $\frac{3}{5}$. Looking at Fig. 6 we can see that $\frac{1}{2}$ of $\frac{3}{5}$ is $\frac{3}{10}$.

We could go on illustrating different types of fractions as we did in the preceding paragraphs, using Figs. 4, 5, and 6, but enough have been offered to demonstrate the principle of **Rule 2.** In multi-

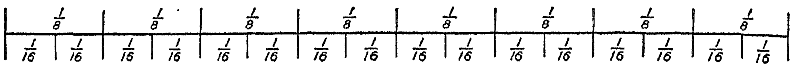


Fig. 5

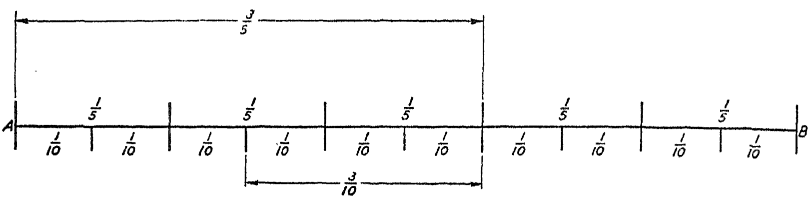


Fig. 6

plying one fraction by another always follow **Rule 2.**

PRACTICE PROBLEMS

In the following problems you are to perform the indicated multiplication. Remember answers must be shown in lowest terms.

1. $\frac{1}{2}\times\frac{1}{5}$ Ans. $\frac{1}{10}$	4. $\frac{5}{10}\times\frac{5}{6}$ Ans. $\frac{5}{12}$
2. $\frac{1}{6}\times\frac{3}{7}$ Ans. $\frac{1}{14}$	5. $\frac{3}{32}\times\frac{8}{10}$ Ans. $\frac{3}{40}$
3. $\frac{2}{3}\times\frac{4}{5}$ Ans. $\frac{8}{15}$	6. $\frac{1}{15}\times\frac{5}{6}$ Ans. $\frac{1}{18}$

There is an easier way of multiplying one fraction by another that can be used in many cases. We can apply the rules of cancellation.

ILLUSTRATIVE EXAMPLES

1. $\frac{2}{3}\times\frac{2}{6}$.

By cancellation $\frac{2}{3}\times\frac{2}{\underset{3}{\cancel{6}}}=\frac{1\times2}{3\times3}=\frac{2}{9}$.

By trial we found that 2 would divide both 2 (of the $\frac{2}{3}$) and 6 (of the $\frac{2}{6}$) an exact number of times. No more cancellation is possible so our answer is $\frac{2}{9}$.

2. $\frac{2}{3}\times\frac{3}{6}$.

$$\text{By cancellation } \frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{1}{\cancel{3}}}{\underset{3}{\cancel{6}}}=\frac{1}{3}.$$

Here we shortened the process after cancellation by leaving out the $\frac{1\times1}{1\times3}$. You will soon learn to do the multiplying without this step.

Note that we are still following **Rule 2.**

3. $\frac{2}{3}\times\frac{4}{6}$.

$$\text{By cancellation } \frac{\overset{1}{\cancel{2}}}{3}\times\frac{4}{\underset{3}{\cancel{6}}}=\frac{4}{9}.$$

4. $\frac{5}{16}\times\frac{2}{5}$.

$$\text{By cancellation } \frac{\overset{1}{\cancel{5}}}{\underset{8}{\cancel{16}}}\times\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{5}}}=\frac{1}{8}.$$

5. $\frac{18}{42}\times$

$$\text{By cancellation } \frac{\overset{\overset{3}{\cancel{9}}}{\cancel{18}}}{\underset{14}{\cancel{42}}}\times\frac{\overset{1}{\cancel{32}}}{\underset{\underset{1}{\cancel{32}}}{\cancel{64}}}=\frac{3}{14}.$$

If you are working with a problem where no cancellation is possible, then simply follow **Rule 2** in the usual manner. But remember that all answers must be shown in lowest terms.

PRACTICE PROBLEMS

1. $\frac{25}{32}\times\frac{2}{3}$	Ans. $\frac{25}{48}$	**6.** $\frac{15}{16}\times\frac{4}{5}$	Ans. $\frac{3}{4}$
2. $\frac{1}{48}\times\frac{24}{25}$	Ans. $\frac{1}{50}$	**7.** $\frac{8}{10}\times\frac{4}{7}$	Ans. $\frac{16}{35}$
3. $\frac{5}{32}\times\frac{4}{5}$	Ans. $\frac{1}{8}$	**8.** $\frac{4}{17}\times\frac{3}{7}$	Ans. $\frac{12}{119}$
4. $\frac{3}{10}\times\frac{5}{8}$	Ans. $\frac{3}{16}$	**9.** $\frac{5}{7}\times\frac{3}{11}$	Ans. $\frac{15}{77}$
5. $\frac{7}{16}\times\frac{4}{5}$	Ans. $\frac{\;}{20}$		

Written Problems. As previously mentioned, written problems require first a careful study to see what is required and then the

application of the proper rule. In the following we will solve a typical written problem to illustrate the reasoning required.

ILLUSTRATIVE EXAMPLE

If you live $\frac{4}{5}$ of a mile from school and have gone $\frac{1}{3}$ of the way, (*a*) how far have you traveled? (*b*) How far do you still have to go?

Solution. It is clear that the school is $\frac{4}{5}$ of a mile from home. This $\frac{4}{5}$ is a distance. Next the problem says that you have gone $\frac{1}{3}$ of the distance from home to school. It is evident that first we must find how far from home you are at $\frac{1}{3}$ of $\frac{4}{5}$ of a mile. We can easily solve this by applying **Rule 2.**

$$\tfrac{1}{3}\times\tfrac{4}{5}=\tfrac{4}{15}\quad \text{Ans. } (a)$$

The next part of the problem asks how far you have to go after having gone $\frac{1}{3}$ of the way. This indicates subtraction. We subtract $\frac{4}{15}$ from $\frac{4}{5}$. Thus we have

$$\tfrac{4}{5}-\tfrac{4}{15}$$

To subtract, they must have the same denominator.

$$\text{L.C.D.}=15$$

$$\tfrac{4}{5}=\tfrac{12}{15}$$

(You learned how to do this in Section 3.)

$$\tfrac{4}{15}=\tfrac{4}{15}$$

$$\text{Then } \tfrac{12}{15}-\tfrac{4}{15}=\tfrac{8}{15}\quad \text{Ans. } (b).$$

Another method: If you have gone $\frac{1}{3}$ of the distance to school, there remains $\frac{3}{3}-\frac{1}{3}$ or $\frac{2}{3}$ of the distance to go. Therefore the remaining distance is $\frac{2}{3}$ of $\frac{4}{5}=\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$ mile. Ans. (*b*).

PRACTICE PROBLEMS

1. John lives $\frac{1}{2}$ of a mile from the post office; Fred lives just half as far as John, and Jim lives $\frac{1}{2}$ as far as Fred. (*a*) How far does Fred live from the post office? (*b*) How far does Jim live from the post office? (*c*) How much farther does John live than Jim? (*a*) Ans. $\frac{1}{4}$ mi. (*b*) Ans. $\frac{1}{8}$ mi. (*c*) Ans. $\frac{3}{8}$ mi.

2. If you had $\frac{3}{4}$ of a pie and ate $\frac{1}{3}$ of what you had, how much of the pie would you have left? Ans. $\frac{1}{2}$ pie.

Improper Fractions. You know what an improper fraction is—it is a fraction in which the numerator is larger than the denominator. For example, $\frac{18}{3}$ is an improper fraction.

It is often necessary to multiply one improper fraction by another improper fraction. To do this, follow **Rule 2** and be sure to use cancellation where possible.

ILLUSTRATIVE EXAMPLES

1. Multiply $\frac{18}{3}$ by $\frac{24}{9}$. Following **Rule 2** and cancelling

$$\frac{\overset{2}{\cancel{18}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{8}{\cancel{24}}}{\underset{1}{\cancel{9}}} = \frac{16}{1} = 16$$

2. Multiply $\frac{11}{7}$ by $\frac{15}{4}$. Following **Rule 2**

$$\frac{11}{7} \times \frac{15}{4} = \frac{165}{28} = 5\frac{25}{28}$$

In Example 1 cancellation was possible whereas in Example 2 it was not. Notice that in each example the answer was given in lowest terms. *It is necessary to give answers in lowest terms in all problems.* Remember this.

Multiplying More Than Two Fractions. Sometimes it is necessary to find the product of a problem such as

$$\frac{1}{4} \times \frac{3}{8} \times \frac{1}{6} \times \frac{2}{3}$$

To solve, follow **Rule 2** and cancel as much as possible.

$$\frac{1}{4} \times \frac{\overset{1}{\cancel{3}}}{8} \times \frac{1}{\underset{3}{\cancel{6}}} \times \frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} = \frac{1 \times 1 \times 1 \times 1}{4 \times 8 \times 3 \times 1} = \frac{1}{96}$$

After cancelling, the second step is the same as in previous examples except that we have more multiplying; the product of the numerator is 1. The denominator is multiplied like this.

$$\begin{array}{r} 4 \times 8 \times 3 \times 1 \\ 4 \\ 8 \\ \hline 32 \\ 3 \\ \hline 96 \\ 1 \\ \hline 96 \end{array}$$

First we multiply 4 by 8. This gives 32. Next multiply 32 by 3. This gives 96. Then $96\times1=96$.

ILLUSTRATIVE EXAMPLES

1. $\frac{4}{10}\times\frac{1}{3}\times\frac{4}{5}\times\frac{6}{7}$.

$$\text{Cancelling } \frac{4}{\underset{5}{\cancel{10}}}\times\frac{1}{\underset{1}{\cancel{3}}}\times\frac{\overset{2}{\cancel{4}}}{5}\times\frac{\overset{2}{\cancel{6}}}{7}=\frac{4\times1\times2\times2}{5\times1\times5\times7}=\frac{16}{175}$$

For the numerator $4 \times 1 \times 2 \times 2$

$$\begin{array}{r} 4 \\ 1 \\ \hline 4 \\ 2 \\ \hline 8 \\ 2 \\ \hline 16 \end{array}$$

For the denominator, second step, we have

$$5 \times 1 \times 5 \times 7$$

$$\begin{array}{r} 5 \\ 1 \\ \hline 5 \\ 5 \\ \hline 25 \\ 7 \\ \hline 175 \end{array}$$

2. $\frac{34}{3}\times\frac{9}{17}\times\frac{15}{2}\times\frac{6}{30}$.

$$\text{Cancelling } \frac{\overset{\overset{1}{\cancel{2}}}{\cancel{34}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{17}}}\times\frac{\overset{1}{\cancel{15}}}{\underset{1}{\cancel{2}}}\times\frac{\overset{3}{\cancel{6}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{30}}}=\frac{1\times3\times1\times3}{1\times1\times1\times1}=\frac{9}{1}=9$$

This example shows that proper and improper fractions can also be multiplied in groups using **Rule 2** and cancellation.

PRACTICE PROBLEMS

The following problems require a knowledge of Lessons 1 and 2. If you have any trouble with any of the problems be sure to go back and review Lessons 1 and 2, because unless you understand these problems thoroughly you should not try Lesson 3.

Note. The answers for the following problems can be found on page 33. Work the problems *first* and then refer to the answers to see how many you worked correctly.

1. Multiply $\frac{3}{7}$ by 5.
2. Multiply $\frac{3}{11}$ by 7.
3. Multiply $\frac{9}{14}$ by 12.
4. Multiply $\frac{5}{21}$ by 63.
5. Multiply $\frac{21}{3}$ by $\frac{27}{7}$.
6. Multiply $\frac{1}{8}$ by $\frac{16}{4}$.
7. Multiply $\frac{18}{4}$ by $\frac{3}{4}$.
8. Multiply $\frac{1}{10}\times\frac{2}{1}\times\frac{3}{4}\times\frac{16}{9}$.
9. Multiply $\frac{81}{121}$ by 8.
10. Multiply $\frac{6}{16}$ by $\frac{9}{24}$.
11. Multiply $\frac{11}{24}$ by $\frac{36}{55}$.
12. Multiply $\frac{208}{416}$ by $\frac{310}{610}$.
13. Multiply 24 by $\frac{7}{8}$.
14. Multiply $\frac{11}{30}$ by 96.
15. Multiply $\frac{5}{144}$ by 48.
16. Find $\frac{15}{16}$ of 320.
17. Find $\frac{13}{25}$ of 375.
18. Find $\frac{23}{50}$ of 600.
19. Find product of $\frac{5}{16}\times\frac{5}{8}$.
20. Find product of $\frac{24}{25}\times\frac{15}{16}$.

Lesson 3

For Step 1, keep in mind what you learned in Lesson 2. For Step 2, learn how to multiply a mixed number by a whole number. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Multiplying a Mixed Number and a Whole Number. Before starting this lesson make sure you remember exactly what a mixed number is. This is explained in Section 3.

Rule 3. *To multiply a mixed number and a whole number, first multiply the fraction (of the mixed number) by the whole number, following* **Rule 1.** *Next multiply the whole numbers; then add the two products.*

ILLUSTRATIVE EXAMPLES

1. Multiply $4\frac{7}{8}$ by 3.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Multiply $\frac{7}{8}$ by 3.	$\frac{7}{8}\times 3=\frac{21}{8}=2\frac{5}{8}$ (Use **Rule 1**)
Step 2	**Step 2**
Multiply 4 by 3.	$4\times 3=12$

Step 3	**Step 3**
Add the two products.	$12+2\frac{5}{8}=14\frac{5}{8}$

In the above solution you will note that in **Step 1** we used **Rule 1** to multiply the fraction by the whole number. Thus we multiplied $\frac{7}{8}\times 3$ just as though there were not another part of the problem still to do. In **Step 2** we multiplied the whole number of the mixed number by 3. In **Step 3** we added the products of **Steps 1** and **2**.

2. Multiply $3\frac{19}{34}$ by 17.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Multiply $\frac{19}{34}$ by 17. Use cancellation method explained in Lesson 1.	$\frac{19}{\underset{2}{\cancel{34}}}\times \overset{1}{\cancel{17}}=\frac{19}{2}=9\frac{1}{2}$
Step 2	**Step 2**
Multiply 3 by 17.	$3\times 17=51$
Step 3	**Step 3**
Add the two results.	$51+9\frac{1}{2}=60\frac{1}{2}$

PRACTICE PROBLEMS

Multiply the following:

1. $81\frac{6}{7}$ by 49	Ans. 4011	**4.** $37\frac{4}{5}$ by 45	Ans. 1701
2. $35\frac{7}{8}$ by 36	Ans. $1291\frac{1}{2}$	**5.** $26\frac{3}{7}$ by 63	Ans. 1665
3. $47\frac{7}{10}$ by 65	Ans. $3100\frac{1}{2}$	**6.** $29\frac{7}{9}$ by 36	Ans. 1072

Lesson 4

For Step 1, keep in mind what you learned in Lesson 3. For Step 2, learn how to multiply a mixed number by a mixed number. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Multiplying Mixed Numbers by Mixed Numbers. You learned how to change a mixed number to an improper fraction in Section 3.

Rule 4. *To multiply one mixed number by another mixed number, change both mixed numbers to improper fractions, then multiply the improper fractions using* **Rule 2.**

ILLUSTRATIVE EXAMPLES

1. $9\frac{1}{3} \times 7\frac{4}{5}$.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Change to improper fractions.	$9\frac{1}{3} = \frac{9 \times 3 + 1}{3} = \frac{28}{3}$ $7\frac{4}{5} = \frac{7 \times 5 + 4}{5} = \frac{39}{5}$
Step 2	**Step 2**
Multiply improper fractions by the cancellation method. (Use **Rule 2.**)	$\frac{28}{\underset{1}{\cancel{3}}} \times \frac{\overset{13}{\cancel{39}}}{5} = \frac{28 \times 13}{1 \times 5} = \frac{364}{5} = 72\frac{4}{5}$

In **Step 1** we changed the $9\frac{1}{3}$ and $7\frac{4}{5}$ to improper fractions. Remember that to change a mixed number to an improper fraction we multiply the whole number by the denominator of the fraction and add the numerator of the fraction to the product. The answer thus obtained is the new numerator over the same denominator as in the original fraction. In **Step 2** we multiplied the two improper fractions using **Rule 2.** We reduced the $\frac{364}{5}$ to simplest terms as we are required to do in any problem.

2. $14\frac{3}{4} \times 16\frac{4}{5}$.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Reduce to improper fractions.	$14\frac{3}{4} = \frac{14 \times 4 + 3}{4} = \frac{59}{4}$ $16\frac{4}{5} = \frac{16 \times 5 + 4}{5} = \frac{84}{5}$
Step 2	**Step 2**
Multiply improper fractions by the cancellation method. (Use **Rule 2**).	$\frac{59}{\cancel{4}} \times \frac{\overset{21}{\cancel{84}}}{5} = \frac{59 \times 21}{1 \times 5} = \frac{1239}{5} = 247\frac{4}{5}$

PRACTICE PROBLEMS

The following problems require a knowledge of Lessons 1 to 4.

After you have worked these problems compare your answers with the answers shown on page 33.

1. Multiply $28\frac{4}{9}$ by $16\frac{3}{4}$.
2. Multiply $38\frac{7}{8}$ by $27\frac{2}{3}$.
3. Multiply $51\frac{4}{5}$ by $72\frac{7}{9}$.
4. Multiply $64\frac{2}{11}$ by $85\frac{3}{5}$.
5. Multiply $12\frac{2}{7}$ by 28.
6. Multiply $3\frac{19}{20}$ by 125.
7. Find the value of $6\frac{3}{5}\times2\frac{3}{11}\times7\frac{8}{9}$.
8. Find the value of $6\frac{3}{7}\times5\frac{4}{5}\times\frac{7}{8}\times2\frac{3}{10}$.
9. Find $5\frac{5}{12}\times\frac{15}{16}$.
10. Find $7\frac{3}{5}\times12$.
11. Find $\frac{81}{121}\times8$.
12. Find $15\frac{5}{8}\times16$.
13. Multiply $\frac{8}{15}$ of $2\frac{1}{4}$ by $\frac{1}{5}$ of $7\frac{1}{3}$.
14. Multiply $\frac{2}{7}$ of 16 by $\frac{7}{10}$ of $26\frac{2}{3}$.
15. A train travels $45\frac{1}{2}$ miles an hour. How far will it go in $7\frac{1}{4}$ hours?
16. A pump puts $147\frac{1}{4}$ gallons of water into a tank in 30 minutes or $\frac{1}{2}$ hour. How many gallons can be put into the tank in $6\frac{3}{4}$ hours?
17. A farmer wants to purchase $2\frac{1}{2}$ tons of hay. He exchanges enough eggs with the owner of the hay to pay for $\frac{1}{4}$ of the hay. How much hay did he have to pay cash for?
18. The trees on one piece of ground yielded $26\frac{2}{3}$ cords of wood. A second piece of ground yielded $2\frac{1}{2}$ times as many cords. How many cords did the second piece of ground yield?
19. A man owned $\frac{2}{3}$ of 123 acres of land, and sold $\frac{1}{2}$ of his share. How many acres did he sell?
20. A mechanic needed 8 pieces of steel rod, each piece $8\frac{3}{4}$ inches long. Neglecting the small amount wasted in sawing, how long must a single rod be in order to provide the 8 pieces needed? Give answer in inches.

Lesson 5

For Step 1, recall cancellation and what you learned in Lesson 1. For Step 2, learn how to divide a fraction by a whole number. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

DIVISION OF FRACTIONS

There are several combinations possible in division of fractions:

1. Dividing a fraction by a whole number

2. Dividing a mixed number by a whole number
3. Dividing one fraction by another fraction
4. Dividing a whole number by a fraction
5. Dividing a mixed number by a mixed number

There is a different rule for each combination, and all are explained in the following.

Dividing a Fraction by a Whole Number. When you studied Section 1 you learned that division is just the opposite of multiplication. In other words you know that if you have 5 apples and multiplied them by 5 you would have 25 apples, or, if you had 5 apples and divided them by 5 you would only have 1 apple. This difference between multiplication and division also applies to fractions.

Remember that, in a fraction, the denominator indicates the number of equal parts into which the whole is divided, and the

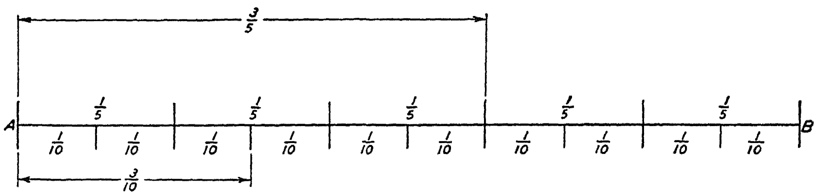


Fig. 7

numerator indicates how many of these parts are being thought of, or used.

Rule 5. *To divide a fraction by a whole number, multiply the denominator of the fraction by the whole number.*

ILLUSTRATIVE EXAMPLES

1. Divide $\frac{3}{5}$ by 2.

Following **Rule 5,**

$$\frac{3}{5} \div 2 = \frac{3}{5 \times 2} = \frac{3}{10}$$

Here we first wrote down the problem with the correct sign between the $\frac{3}{5}$ and the 2. Next we followed the rule and multiplied the denominator (5) by 2. Thus 5×2 equals 10. The 10 becomes the denominator of the answer. The numerator of the answer is the same as in the original fraction before division took place.

We can illustrate this process of division by a simple experiment. Note Fig. 7. In the space above line AB we divided the

line into 5 equal parts. Thus each part is $\frac{1}{5}$ of the line. In the space below line AB we divided the line into 10 equal parts so that each part is $\frac{1}{10}$ of line AB. We know that if we divide anything by 2 we are dividing it into two parts. If we divide anything by 3 we are really dividing it into 3 parts, etc. Now, using our example, $\frac{3}{5}\div 2$, and looking at Fig. 7, we can see that $\frac{3}{5}\div 2$ equals $\frac{3}{10}$.

2. Divide $\frac{4}{5}$ by 4.

Following **Rule 5,**

$$\frac{4}{5}\div 4=\frac{4}{5\times 4}=\frac{4}{20}=\frac{1}{5}$$

We multiplied the denominator of the fraction by the whole number and kept the same numerator. The $\frac{4}{20}$ is not in lowest terms. It is reduced to $\frac{1}{5}$.

Looking at Fig. 7 we can see that if 4 of the $\frac{1}{5}$ spaces are taken as a whole and then divided by 4, we have 1 of the $\frac{1}{5}$ spaces.

3. Divide $\frac{7}{8}$ by 8.

Following **Rule 5,**

$$\frac{7}{8}\div 8=\frac{7}{8\times 8}=\frac{7}{64}$$

PRACTICE PROBLEMS

1. $\frac{30}{32}\div 5$ Ans. $\frac{3}{16}$
2. $\frac{33}{35}\div 11$ Ans. $\frac{3}{35}$
3. $\frac{141}{108}\div 3$ Ans. $\frac{47}{108}$
4. $\frac{39}{51}\div 7$ Ans. $\frac{13}{119}$
5. $\frac{37}{20}\div 4$ Ans. $\frac{37}{80}$
6. $\frac{220}{99}\div 5$ Ans. $\frac{4}{9}$
7. $\frac{47}{8}\div 8$ Ans. $\frac{47}{64}$
8. $\frac{23}{15}\div 6$ Ans. $\frac{23}{90}$
9. $\frac{65}{10}\div 13$ Ans. $\frac{1}{2}$

Lesson 6

For Step 1, recall what you learned in Lesson 3. For Step 2, learn how to divide a mixed number by a whole number. For Step 3, work the Illustrative Examples. Study the solutions until you understand them, then lay the book aside and work the examples without looking at the solutions. For Step 4, work the Practice Problems.

Dividing a Mixed Number by a Whole Number. This process is only a little more complicated than the previous process of dividing fractions by whole numbers.

Rule 6. *To divide a mixed number by a whole number, change the mixed number to an improper fraction and then do the division as explained in* **Rule 5.**

ILLUSTRATIVE EXAMPLES

1. $2\frac{7}{8} \div 3.$

Solution

Instruction	*Operation*
Step 1 Change mixed numbers to improper fraction as taught in Section 3.	**Step 1** $2\frac{7}{8} = \frac{2\times8+7}{8} = \frac{23}{8}$
Step 2 Divide the $\frac{23}{8}$ by 3, using the process of **Rule 5.**	**Step 2** $\frac{23}{8} \div 3 = \frac{23}{8\times3} = \frac{23}{24}$ Ans.

2. $4\frac{2}{9} \div 5$

Solution

Instruction	*Operation*
Step 1 Reduce to improper fraction.	**Step 1** $= \frac{4\times9+2}{9} = \frac{38}{9}$
Step 2 Divide by 5 by multiplying denominator by 5 (**Rule 5**).	**Step 2** $\frac{38}{9} \div 5 = \frac{38}{45}$ Ans.

PRACTICE PROBLEMS

The following problems require a knowledge of Lessons 5 and 6.

After you have worked the problems compare them with the answers shown on page 33.

1. $250\frac{1}{2} \div 5.$
2. $333\frac{1}{3} \div 3.$
3. $293\frac{1}{6} \div 11.$
4. $789\frac{2}{5} \div 4.$
5. $877\frac{1}{7} \div 7.$
6. $724\frac{2}{9} \div 8.$
7. $\frac{13}{17} \div 14.$
8. $\frac{14}{15} \div 5.$
9. $\frac{9}{21} \div 3.$
10. $2\frac{4}{7} \div 6.$
11. $2\frac{22}{25} \div 36.$
12. $5\frac{13}{25} \div 23.$
13. $17\frac{7}{9} \div 8.$

14. If a horse eats $\frac{9}{10}$ of a ton of hay in 3 months, what part of a ton will last him 1 month?

15. A grocer had $32\frac{1}{2}$ pounds of a staple goods which he decided to divide equally into 8 bags. How many pounds did he put into each bag?

16. A car owner drove $121\frac{3}{4}$ miles on 10 gallons of gasoline. How many miles per gallon did he obtain from the gasoline?

17. A farmer found that in order to run a single wire around a piece of land it required 110 feet of wire. If he had $329\frac{3}{5}$ feet of wire, how many wires could be put on a fence around this piece of land?

18. If you had $\frac{4}{5}$ of a pie and wanted to serve each of 4 people with an equal portion, how much would you give each?

19. If a garage man had $4\frac{2}{3}$ gallons of anti-freeze which he wanted to divide between 7 autos, how much would he put in each auto?

20. How many times will $16\frac{3}{4}$ gallons of cider fill a vessel that holds 3 gallons?

Lesson 7

For Step 1, recall what you learned in Lesson 2, learn how to divide a fraction by a fraction. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Dividing a Fraction by a Fraction. In order to divide one fraction by another fraction we must use an entirely new principle. However, it is very easy and will not cause you any trouble.

Rule 7. *To divide one fraction by another fraction, invert the divisor fraction and multiply as learned for* **Rule 2.**

If we have a problem such as $\frac{2}{3} \div \frac{1}{2}$, the $\frac{1}{2}$ is the divisor fraction. To invert a fraction we turn it upside down. Thus $\frac{1}{2}$ inverted becomes $\frac{2}{1}$.

ILLUSTRATIVE EXAMPLES

1. Divide $\frac{2}{3}$ by $\frac{3}{4}$.

Solution

Instruction	*Operation*
Step 1 Invert the divisor.	**Step 1** $\frac{3}{4}$ inverted is $\frac{4}{3}$
Step 2 Multiply the fractions as explained for **Rule 2.**	**Step 2** $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$ Ans.

2. Divide $\frac{7}{8}$ by $\frac{2}{5}$.

Solution

Instruction	*Operation*
Step 1 Invert the divisor.	**Step 1** $\frac{2}{5}$ inverted is $\frac{5}{2}$
Step 2 Multiply fractions as explained for **Rule 2.**	**Step 2** $\frac{7}{8} \times \frac{5}{2} = \frac{35}{16} = 2\frac{3}{16}$ Ans.

Sometimes we can shorten the multiplication step by cancellation.

3. Divide $\frac{7}{9}$ by $\frac{2}{3}$.

Solution

Instruction	*Operation*
Step 1 Invert the divisor.	**Step 1** $\frac{2}{3}$ inverted is
Step 2 Cancel and then multiply as explained for **Rule 2.**	**Step 2** $\frac{7}{9}\times\frac{3}{2}=\frac{7}{\underset{3}{\cancel{9}}}\times\frac{\cancel{3}}{2}=\frac{7}{6}=1\frac{1}{6}$ Ans.

PRACTICE PROBLEMS

1. $\frac{13}{15}\div\frac{2}{3}$	Ans. $1\frac{3}{10}$	4. $\frac{14}{17}\div\frac{28}{34}$	Ans. 1
2. $\frac{25}{35}\div\frac{1}{4}$	Ans. $2\frac{6}{7}$	5. $\frac{350}{475}\div\frac{50}{75}$	Ans. $1\frac{2}{19}$
3. $\frac{11}{15}\div\frac{13}{27}$	Ans. $1\frac{34}{65}$	6. $\frac{257}{258}\div\frac{7}{9}$	Ans. $1\frac{169}{602}$

Lesson 8

For Step 1, recall what you learned in Lesson 1 and in Lesson 5. For Step 2, learn how to divide a whole number by a fraction. For Step 3, work the Illustrative Examples. In order to obtain the maximum benefit from the Illustrative Examples, study the solutions until you understand every step thoroughly. Then lay the text to one side and see if you can work each of the examples without looking at the solutions. If you experience any trouble in solving them you should review the entire lesson preceding the examples. For Step 4, work the Practice Problems.

Dividing a Whole Number by a Fraction. We have one more new principle to learn for this operation. In dividing a whole number by a fraction we write the whole number in the form of a fraction. Take the whole number 10. If this is written in the form of a fraction it is $\frac{10}{1}$. Remember that whenever you change a whole number to the fraction form, the whole number must be the numerator and that the denominator must be 1.

Rule 8. *To divide a whole number by a fraction, change the whole number into fraction form, invert the divisor fraction, and multiply.*

ILLUSTRATIVE EXAMPLES

1. Divide 10 by $\frac{2}{5}$.

Solution

Instruction

Step 1

Change the whole number to the form of a fraction.

Step 2

Invert the divisor.

Step 3

Multiply fractions as explained in **Rule 2**. Cancel where possible.

Operation

Step 1

10 in fraction form is $\frac{10}{1}$

Step 2

$\frac{2}{5}$ inverted is $\frac{5}{2}$

Step 3

$$\frac{\overset{5}{\cancel{10}}}{1}\times\frac{5}{\underset{1}{\cancel{2}}}=\frac{25}{1}=25 \text{ Ans.}$$

2. Divide 36 by $\frac{9}{13}$.

Solution

Instruction

Step 1

Change the whole number to the form of a fraction.

Step 2

Invert divisor.

Step 3

Multiply the fractions as explained for **Rule 2**. Cancel where possible.

Operation

Step 1

36 in fractional form is $\frac{36}{1}$

Step 2

$\frac{9}{13}$ inverted is $\frac{13}{9}$

Step 3

$$\frac{\overset{4}{\cancel{36}}}{1}\times\frac{13}{\cancel{9}}=\frac{52}{1}=52 \text{ Ans.}$$

PRACTICE PROBLEMS

1. $25\div\frac{5}{7}$	Ans. 35	**4.** $65\div\frac{13}{15}$	Ans. 75
2. $42\div\frac{6}{7}$	Ans. 49	**5.** $81\div\frac{9}{11}$	Ans. 99
3. $26\div\frac{4}{9}$	Ans. $58\frac{1}{2}$	**6.** $99\div\frac{11}{13}$	Ans. 117

Lesson 9

For Step 1, recall what you learned in Lesson 4. For Step 2, learn how to divide a mixed number by a mixed number. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Dividing a Mixed Number by a Mixed Number. No new principles are required here. We simply make use of some things we have already learned.

Rule 9. *To divide a mixed number by a mixed number, change both mixed numbers to improper fractions, invert the divisor fraction, and multiply.*

ILLUSTRATIVE EXAMPLES

1. $7\frac{2}{9} \div 4\frac{1}{3}$.

Solution

Instruction	*Operation*
Step 1 Change to improper fractions.	**Step 1** $7\frac{2}{9} = \frac{7 \times 9 + 2}{9} = \frac{65}{9}$ $4\frac{1}{3} = \frac{4 \times 3 + 1}{3} = \frac{13}{3}$
Step 2 Invert divisor, use cancellation, and multiply.	**Step 2** $\frac{\overset{5}{\cancel{65}}}{\underset{3}{\cancel{9}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{13}}} = \frac{5}{3}$
Step 3 Reduce.	**Step 3** $\frac{5}{3} = 1\frac{2}{3}$ Ans.

2. $8\frac{3}{4} \div 7\frac{1}{2}$.

Step 1 Change to improper fractions.	**Step 1** $8\frac{3}{4} = \frac{8 \times 4 + 3}{4} = \frac{35}{4}$ $7\frac{1}{2} = \frac{7 \times 2 + 1}{2} = \frac{15}{2}$
Step 2 Invert the divisor, use cancellation, and multiply, then reduce to lowest terms.	**Step 2** $\frac{\overset{7}{\cancel{35}}}{\underset{2}{\cancel{4}}} \times \frac{\overset{1}{\cancel{2}}}{\underset{3}{\cancel{15}}} = \frac{7}{6} = 1\frac{1}{6}$

3. $15\frac{4}{7} \div 21\frac{2}{9}$.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Reduce to improper fractions.	$15\frac{4}{7}=\frac{15\times7+4}{7}=\frac{109}{7}$ $21\frac{2}{9}=\frac{21\times9+2}{9}=\frac{191}{9}$
Step 2	**Step 2**
Invert the divisor and multiply.	$\frac{109}{7}\times\frac{9}{191}=\frac{981}{1337}$

PRACTICE PROBLEMS

For the following problems a knowledge of Lessons 7, 8, and 9 is necessary. *After* you have solved all of the problems, check your answers with those shown on page 33.

1. $4\frac{2}{3}\div3\frac{1}{2}$.
2. $7\frac{1}{5}\div6\frac{1}{4}$.
3. $6\frac{1}{2}\div5\frac{1}{4}$.
4. $8\frac{2}{5}\div7\frac{1}{10}$.
5. $8\frac{1}{9}\div6\frac{2}{3}$.
6. $7\frac{1}{8}\div5\frac{3}{4}$.
7. $42\div\frac{6}{7}$.

   $\div\frac{[illegible]}{12}$.
9. $316\div\frac{9}{25}$.
10. $7\div\frac{3}{4}$.
11. $\frac{2}{3}\div\frac{27}{28}$.
12. $48\div\frac{15}{16}$.
13. $\frac{133}{144}\div\frac{19}{24}$.

14. If a carpenter had a board $13\frac{1}{2}$ feet long, how many pieces each $2\frac{1}{4}$ feet long could he cut out of it, neglecting the waste caused by sawing?

15. If a ball of fish line contains 324 feet of line how many pieces of line $20\frac{1}{4}$ feet long can this ball be divided into?

16. A baker has a mixer full of bread dough which weighs 210 pounds. First he takes enough of this dough to make 160 one-pound loaves. How many $1\frac{1}{4}$-pound loaves can be made from the balance?

17. How many times will $46\frac{3}{4}$ gallons of oil fill a can that holds $3\frac{1}{4}$ gallons?

18. A family spent $\frac{3}{20}$ of their income for clothing and $\frac{1}{4}$ for groceries. The bills for the two items total \$650. What was the income?

19. A rug salesman sold $\frac{2}{5}$ of his stock in one week and $\frac{1}{2}$ of what was left the next week. How much stock was still on hand?

20. A ranchman had 322 sheep. He shipped $\frac{2}{7}$ of his flock to Omaha. He also shipped $\frac{3}{5}$ of the remainder to Chicago. How many sheep did the ranchman have left?

Lesson 10

For Step 1, recall all you have had in the preceding lessons. For Step 2, learn how to detect complex fractions and how to operate with them. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Complex Fractions

A **complex fraction** is one that has its numerator, denominator, or both, composed of fractions, improper fractions, or mixed numbers.

Case (A) $\dfrac{\frac{1}{2}}{4}$ (numerator) (denominator)

Here the numerator is a fraction and the denominator is the same as though we had written $\frac{1}{4}$. You must learn to think of the $\frac{1}{2}$ as the numerator. As the complex fraction stands, it means that in order to express its value in simplest terms we must divide $\frac{1}{2}$ by 4. The line between the $\frac{1}{2}$ and the 4 thus represents division. To reduce such a complex fraction to lowest terms we always divide the numerator by the denominator.

$$\overset{(a)}{\frac{\frac{1}{2}}{4}} = \overset{(b)}{\tfrac{1}{2} \div 4} = \overset{(c)}{\frac{1}{2\times 4}} = \overset{(d)}{\tfrac{1}{8}}$$

Explanation of solution.

(*a*) The complex fraction which indicates $\frac{1}{2}$ must be divided by 4.

(*b*) The problem expressed in usual division.

(*c*) Here, following **Rule 5,** we multiplied the denominator of the fraction by the whole number.

(*d*) Answer.

From this solution you can readily see that you must thoroughly understand the nine Rules given in Lessons 1 through 9. So if you did not understand any of the above steps be sure to review **Rules 1 to 9.**

PRACTICE PROBLEMS

Reduce the following complex fractions to simplest and lowest terms.

1. $\dfrac{\frac{1}{10}}{3}$ Ans. $\frac{1}{30}$ **3.** $\dfrac{\frac{17}{25}}{34}$ Ans. $\frac{1}{50}$ **5.** $\dfrac{\frac{25}{5}}{50}$ Ans. $\frac{1}{10}$

2. Ans. $\frac{3}{8}$ **4.** $\dfrac{\frac{10}{8}}{10}$ Ans. $\frac{1}{8}$

Case (B) $\dfrac{3}{\frac{2}{3}}$ (numerator) (denominator)

Here the numerator is a whole number. The denominator is a fraction. As this complex fraction stands, it means that in order to express its value in simplest terms we must divide 3 by $\frac{2}{3}$. As in **Case** A the line indicates division. We always have to divide the

numerator of a fraction by its denominator in order to reduce it to lowest terms.

$$\begin{array}{ccccc} (a) & (b) & (c) & (d) & (e) \end{array}$$

$$\frac{3}{\frac{2}{3}}=3\div\frac{2}{3}=\frac{3}{1}\div\frac{2}{3}=\frac{3}{1}\times\frac{3}{2}=\frac{9}{2}=4\frac{1}{2}$$

Explanation of solution.

(*a*) The complex fraction which indicates 3 must be divided by $\frac{2}{3}$.

(*b*) The problem expressed in usual division.

(*c*) Here, following **Rule 8,** we changed the whole number to fractional form.

(*d*) Here we followed **Rule 7** and inverted the divisor fraction. No cancellation was possible so we multiplied.

(*e*) The answer $\frac{9}{2}$ is an improper fraction so it must be reduced to a mixed number. The final answer, $4\frac{1}{2}$, thus represents our problem expressed in simplest and lowest terms.

PRACTICE PROBLEMS

Reduce the following complex fractions to simplest and lowest terms.

1. $\frac{5}{\frac{5}{6}}$ Ans. 6
2. $\frac{10}{\frac{7}{8}}$ Ans. $11\frac{3}{7}$
3. $\frac{15}{\frac{10}{3}}$ Ans. $4\frac{1}{2}$
4. $\frac{30}{\frac{4}{30}}$ Ans. 225
5. $\frac{16}{\frac{1}{4}}$ Ans. 64

Case (C) $\dfrac{1\frac{2}{3}\ \text{(numerator)}}{3\frac{1}{2}\ \text{(denominator)}}$

Here both numerator and denominator are mixed numbers. The line between means that the numerator must be divided by the denominator in order to reduce the complex fraction to simplest terms.

$$\begin{array}{ccccc} (a) & (b) & (c) & (d) & (e) \end{array}$$

$$\frac{1\frac{2}{3}}{3\frac{1}{2}}=1\frac{2}{3}\div3\frac{1}{2}=\frac{5}{3}\div\frac{7}{2}=\frac{5}{3}\times\frac{2}{7}=\frac{10}{21}$$

Explanation of solution.

(*a*) The complex fraction which indicates $1\frac{2}{3}$ must be divided by $3\frac{1}{2}$.

(*b*) The problem expressed in usual division.

(*c*) Mixed numbers to improper fractions. **Rule 9.**

(*d*) Here we followed **Rule 7** and inverted the divisor fraction. No cancellation was possible so we multiplied.

(*e*) Answer.

Case (D) $\dfrac{\frac{3}{4}+\frac{1}{2}}{\frac{2}{3}+\frac{5}{6}}$ (numerator) (denominator)

Here we have a more complicated complex fraction. However, it is solved just as easily as in **Cases A, B,** and **C.** As in the previous cases, the line indicates division. And, as we previously learned, in order to simplify this complex fraction we have to divide the numerator by the denominator. But before we can do any division with the numerator and denominator we have to simplify them. In other words we have to add the two fractions in the numerator and the two in the denominator. Then we can solve the problem as in **Case C.**

(*a*) (*b*) (*c*) (*d*) (*e*)

$$\frac{\frac{3}{4}+\frac{1}{2}}{\frac{2}{3}+\frac{5}{6}}=\frac{\frac{5}{4}}{\frac{9}{6}}=\frac{5}{4}\div\frac{9}{6}=\frac{5}{\cancel{4}_{2}}\times\frac{\cancel{6}^{3}}{9}=\frac{\cancel{15}^{5}}{\cancel{18}_{6}}=\frac{5}{6}$$

Explanation of solution.

(*a*) The complex fraction which indicates that $\frac{3}{4}+\frac{1}{2}$ must be divided by $\frac{2}{3}+\frac{5}{6}$.

(*b*) Here we added $\frac{3}{4}$ and $\frac{1}{2}$ and got $\frac{5}{4}$. Also we added $\frac{2}{3}$ and $\frac{5}{6}$ and got $\frac{9}{6}$. This is called combining fractions.

Note: You learned how to add fractions in Section 3. If this step of the problem is not clear, be sure to go back and study L.C.D. and Addition of Fractions in Section 3.

(*c*) Here we have the problem after we have combined (added) the fractions in the numerator and those in the denominator. The problem is shown here in regular division form.

(*d*) Here we followed **Rule 7** and inverted the divisor fraction. Some cancellation was possible, after which we multiplied.

(*e*) The answer $\frac{15}{18}$ is not in its lowest terms so we reduced it to $\frac{5}{6}$.

Case (E) $\dfrac{5\frac{1}{2}-\frac{1}{3}}{\dfrac{1}{1\frac{1}{3}}+2\frac{1}{4}}$ (numerator) (denominator)

In this problem we have a much more complicated complex fraction. However, its reduction to simplest and lowest terms is easy if we do it carefully.

In this case the numerator is $5\frac{1}{2}-\frac{1}{3}$ and we can see that we have a mixed number minus a fraction. To simplify this numerator change the $5\frac{1}{2}$ to an improper fraction and subtract $\frac{1}{3}$ from it.

The denominator is $\dfrac{1}{1\frac{1}{3}}+2\frac{1}{4}$ and we see a fraction with a mixed number denominator plus a mixed number. To simplify this denominator we have to change the $1\frac{1}{3}$ to an improper fraction and divide it into 1. To this quotient we must add $2\frac{1}{4}$.

(a) (b) (c) (d) (e) (f) (g)

$$\frac{5\frac{1}{2}-\frac{1}{3}}{\dfrac{1}{1\frac{1}{3}}+2\frac{1}{4}}=\frac{\frac{11}{2}-\frac{1}{3}}{\dfrac{1}{\frac{4}{3}}+\frac{9}{4}}=\frac{\frac{11}{2}-\frac{1}{3}}{\frac{3}{4}+\frac{9}{4}}=\frac{\frac{31}{6}}{\frac{12}{4}}=\frac{31}{6}\div\frac{12}{4}=\frac{31}{\cancel{6}_{3}}\times\frac{\cancel{4}^{\cancel{2}^{1}}}{\cancel{12}_{6}}=\frac{31}{18}=1\frac{13}{18}$$

Explanation of solution.

(*a*) The complex fraction which indicates that $5\frac{1}{2}-\frac{1}{3}$ must be divided by $\dfrac{1}{1\frac{1}{3}}+2\frac{1}{4}$.

(*b*) In this step the $5\frac{1}{2}$, $1\frac{1}{3}$, and $2\frac{1}{4}$ were all changed to improper fractions. This is necessary because we cannot do anything with mixed numbers until they have been changed to improper fractions.

(*c*) Here we followed **Rule 8** and simplified the $\dfrac{1}{\frac{4}{3}}$. This, when written in regular division form is $1\div\frac{4}{3}$. Then, as explained in **Rule 8**, $1\div\frac{4}{3}=\frac{1}{1}\div\frac{4}{3}=\frac{1}{1}\times\frac{3}{4}=\frac{3}{4}$. This $\frac{3}{4}$ then replaces $\dfrac{1}{\frac{4}{3}}$.

(*d*) Here we subtracted $\frac{1}{3}$ from $\frac{11}{2}$ and added $\frac{3}{4}$ to $\frac{9}{4}$. Remember that you learned how to find L.C.D. and add and subtract fractions in Section 3.

(*e*) Here the problem, partly simplified, is written in regular division form.

(*f*) We followed **Rule 7** and inverted the divisor fraction. Some cancellation was possible, after which we multiplied.

(*g*) The answer $\frac{31}{18}$ is an improper fraction so we reduced it to $1\frac{13}{18}$ which is the answer in lowest terms.

Case (F) $\dfrac{(\frac{5}{8}-\frac{1}{2})+(\frac{5}{6}\times\frac{1}{4})}{1\frac{3}{4}\times(\frac{1}{2}+\frac{5}{6})}$ numerator / denominator

In this case we have *parentheses* added to increase the complication of the complex fraction, but it is still an easy problem to solve when we do one thing at a time.

Rule 10. *When simplifying a complex fraction which contains parenthesis, always do what is indicated within the parenthesis first and this removes the parenthesis.*

Thus the $(\frac{5}{8}-\frac{1}{2})$ indicates that we must subtract $\frac{1}{2}$ from $\frac{5}{8}$. When we do this we remove the parenthesis.

(*a*) (*b*) (*c*) (*d*) (*e*) (*f*) (*g*) (*h*)

$$\frac{(\frac{5}{8}-\frac{1}{2})+(\frac{5}{6}\times\frac{1}{4})}{1\frac{3}{4}\times(\frac{1}{2}+\frac{5}{6})}=\frac{\frac{1}{8}+\frac{5}{24}}{1\frac{3}{4}\times\frac{8}{6}}=\frac{\frac{1}{8}+\frac{5}{24}}{\frac{7}{4}\times\frac{8}{6}}=\frac{\frac{8}{24}}{\frac{7}{4}\times\frac{8}{6}}=\frac{\frac{8}{24}}{\frac{7}{3}}=\frac{8}{24}\div\frac{7}{3}=\frac{\overset{1}{\cancel{8}}}{\underset{\cancel{8}}{\cancel{24}}}\times\frac{\overset{1}{\cancel{3}}}{7}=\frac{1}{7}$$

(*a*) The problem.

(*b*) In this step, following **Rule 10**, we eliminated all parentheses as follows:

First we subtracted $\frac{1}{2}$ from $\frac{5}{8}$ and got $\frac{1}{8}$. We wrote the $\frac{1}{8}$ in place of $(\frac{5}{8}-\frac{1}{2})$. You learned how to find L.C.D. and add and subtract fractions in Section 3.

Next we followed **Rule 2** and multiplied $\frac{5}{6}\times\frac{1}{4}$. This gave us $\frac{5}{24}$. We wrote $\frac{5}{24}$ in place of $(\frac{5}{6}\times\frac{1}{4})$.

Next we added $\frac{1}{2}+\frac{5}{6}$ and got $\frac{8}{6}$. Then we wrote $\frac{8}{6}$ in place of $(\frac{1}{2}+\frac{5}{6})$.

(*c*) Here we changed the $1\frac{3}{4}$ to an improper fraction.

(*d*) Here we added $\frac{1}{8}+\frac{5}{24}$ and got $\frac{8}{24}$.

(*e*) Here we followed **Rule 2** and multiplied $\frac{7}{4}$ by $\frac{8}{6}$ and got $\frac{56}{24}$ or $\frac{7}{3}$.

(*f*) Here we have written the problem in regular division form.

(*g*) Here we followed **Rule 7** and inverted the divisor fraction. Cancellation was followed by multiplication.

(*h*) Answer.

SPECIAL NOTICE

No student can learn how to apply **Rules 1 to 10** or how to solve complex fractions unless he first studies the illustrative examples and then tests his knowledge by seeing if he can work each illustrative example without looking at the text. You may think you understand these examples after you have read them. However, the only way you can be sure you understand them is by actually working them figure for figure. This is especially true of complex fractions.

PRACTICE PROBLEMS

After you have worked the following problems, check your answers with the answers shown on page 33.

Simplify the following. Give answers in lowest terms.

1. $\dfrac{\frac{13}{16}}{19}$

2. $\dfrac{2\frac{1}{4}}{\frac{5}{6}}$

3. $\dfrac{\frac{3}{4}}{\frac{5}{8}}$

4. $\dfrac{\frac{2}{3}}{\frac{8}{9}}$

5. $\dfrac{3\frac{1}{2}}{2\frac{1}{3}}$

6. $\dfrac{24}{\frac{5}{16}}$

7. $\dfrac{3\frac{1}{2}}{4\frac{3}{8}}$

8. $\dfrac{6\frac{2}{9}}{8\frac{2}{3}}$

9. $\dfrac{11\frac{3}{7}}{\frac{4}{7}}$

10. $\dfrac{\frac{5}{11}}{4\frac{2}{5}}$

11. $\dfrac{\frac{2}{5}\times\frac{5}{6}}{\frac{2}{9}\times 4\frac{1}{2}}$

12. $\frac{\text{[illegible]}}{21}$

13. $\dfrac{16\frac{2}{3}}{33\frac{1}{3}}$

14. $\dfrac{5\frac{1}{2}+(\frac{3}{4}\div\frac{1}{2})}{(2+\frac{2}{3})\div\frac{3}{5}}$

15. If a horse eats $\frac{3}{8}$ of a bushel of oats in a day, in how many days will he eat $5\frac{1}{4}$ bushels?

16. How many times will $4\frac{3}{8}$ gallons of oil fill a vessel that holds $\frac{1}{2}$ of 1 gallon?

17. If 14 acres of meadow land produce $32\frac{2}{3}$ tons of hay, how many tons will 5 acres produce?

ANSWERS TO PRACTICE PROBLEMS

Lesson 2, Page 15

1. $2\frac{1}{7}$. **2.** $1\frac{10}{11}$. **3.** $7\frac{5}{7}$. **4.** 15. **5.** 27. **6.** $\frac{1}{2}$. **7.** . **8.** $\frac{4}{15}$. **9.** $5\frac{43}{121}$. **10.** $\frac{9}{64}$. **11.** $\frac{3}{10}$. **12.** $\frac{31}{122}$. **13.** 21. **14.** $35\frac{1}{5}$. **15.** $1\frac{2}{3}$ **16.** 300. **17.** 195. **18.** 276. **19.** $\frac{25}{128}$. **20.** $\frac{9}{10}$.

Lesson 4, Page 18

1. $476\frac{4}{9}$. **2.** $1075\frac{13}{24}$. **3.** $3769\frac{8}{9}$. **4.** $5493\frac{53}{55}$. **5.** 344. **6.** $493\frac{3}{4}$. **7.** $118\frac{1}{3}$. **8.** $75\frac{3}{80}$. **9.** $5\frac{5}{64}$. **10.** $91\frac{1}{5}$. **11.** $5\frac{43}{121}$. **12.** 250. **13.** $1\frac{19}{25}$. **14.** $85\frac{1}{3}$. **15.** $329\frac{7}{8}$ miles. **16.** $1987\frac{7}{8}$ gallons. **17.** $1\frac{7}{8}$ tons. **18.** $66\frac{2}{3}$ cords. **19.** 41 acres. **20.** 70 inches.

Lesson 6, Page 21

1. $50\frac{1}{10}$. **2.** $111\frac{1}{9}$. **3.** $26\frac{43}{66}$. **4.** $197\frac{7}{20}$. **5.** $125\frac{15}{49}$. **6.** $90\frac{19}{36}$. **7.** $\frac{13}{238}$. **8.** $\frac{14}{75}$. **9.** $\frac{1}{7}$. **10.** $\frac{3}{7}$. **11.** $\frac{2}{25}$. **12.** $\frac{6}{25}$. **13.** $2\frac{2}{9}$. **14.** $\frac{3}{10}$ ton. **15.** $4\frac{1}{16}$ pounds. **16.** $12\frac{7}{40}$ miles per gallon. **17.** 3 wires. **18.** $\frac{1}{5}$ pie. **19.** $\frac{2}{3}$ gallon. **20.** $5\frac{7}{12}$ times.

Lesson 9, Page 26

1. $1\frac{1}{3}$. **2.** $1\frac{1}{1}$ **3.** $1\frac{5}{21}$. **4.** $1\frac{13}{71}$. **5.** $1\frac{13}{60}$. **6.** $1\frac{11}{46}$. **7.** 49. **8.** $205\frac{5}{7}$. **9.** $877\frac{7}{9}$. **10.** $1\frac{1}{6}$. **11.** **12.** $51\frac{1}{5}$. **13.** $1\frac{1}{6}$. **14.** 6 pieces. **15.** 16 pieces. **16.** 40 loaves. **17.** $14\frac{5}{13}$ times. **18.** $1,625. **19.** $\frac{3}{10}$. **20.** 92 sheep.

Lesson 10, Page 32

1. $\frac{13}{304}$. **2.** $2\frac{7}{10}$. **3.** $1\frac{1}{5}$. **4.** $\frac{3}{4}$. **5.** $1\frac{1}{2}$. **6.** $76\frac{4}{5}$. **7.** $\frac{4}{5}$. **8.** $\frac{28}{39}$. **9.** 20. **10.** $\frac{25}{242}$. **11.** $\frac{1}{3}$. **12.** $3\frac{27}{40}$. **13.** $\frac{1}{2}$. **14.** $1\frac{23}{40}$. **15.** 14 days. **16.** $8\frac{3}{4}$ times. **17.** $11\frac{2}{3}$ tons.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems check your answers with the solutions shown on page 36.

If you miss more than two of the problems, it means that you should review the whole book very carefully.

Do not try this trial examination until you have worked every Practice Problem in the book.

1. Multiply (a) $\frac{3}{4}\times\frac{16}{3}\times\frac{18}{24}\times\frac{12}{9}$.

(b) $\frac{7}{32}\times16\times3\frac{1}{4}$.

2. Divide (a) $\frac{13}{10}$ by $6\frac{1}{4}$.

(b) 39 by $\frac{3}{2}$.

3. Reduce to simplest form and lowest terms.

$$\frac{\frac{4}{\frac{1}{2}}\times\frac{1}{3}}{(\frac{1}{2}+\frac{4}{2})\times(\frac{2}{3}-\frac{1}{3})}$$

4. If the minuend is $46\frac{4}{5}$ and the remainder is $26\frac{4}{5}$ what is the subtrahend?

HINT: Remember what you learned in Section 1. Write this problem down as you learned from subtraction.

5. If you are $18\frac{1}{3}$ years old and your father has lived $2\frac{1}{2}$ times as long, what is your father's age?

6. If a steel railroad rail weighs $110\frac{1}{2}$ pounds a foot how much will $\frac{1}{2}$ foot weigh?

7. A shopper bought $2\frac{1}{3}$ pounds of frosted cookies, 2 pounds of plain cookies, and $\frac{1}{3}$ pound of chocolate cookies. What was her bill in dollars and fractions of dollars if all cookies cost $\frac{1}{3}$ dollar a pound?

8. Multiply $2\frac{6}{10}$ by 3 and divide that product by $\frac{2}{3}\times\frac{6}{8}$.

9. Divide $\frac{2}{3}$ of 24 by the product of $2\frac{1}{8}\times\frac{3}{2}$.

10. Multiply $\frac{6}{10}\times\frac{24}{18}\times\frac{3}{36}\times\frac{48}{24}$ and divide this product by $\frac{\frac{1}{3}}{2\frac{3}{5}}$.

If you have followed directions in studying this text; if you have worked out the Practice Problems in each lesson; and if you have sent to the School for information and help on the problems which you could not get, you are ready for the examination.

If you have not worked out the Practice Problems, do not start the examination, for you may fail to get a passing grade. It is by working the Practice Problems that you get an understanding of each point presented in the book. The examination is merely a test of your general knowledge of the subject.

It is for your own good that we are emphasizing the importance of learning each lesson. If you fail to learn each lesson thoroughly, you will find difficulty not only in understanding advanced mathematics, wherein these principles are applied, but also in solving the problems encountered in your daily work.

FINAL EXAMINATION

Use cancellation when possible.

1. Divide (a) $4\frac{2}{3}$ by 2 (b) $\frac{12}{3}$ by $3\frac{1}{3}$
2. Multiply (a) $\frac{7}{8}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{7}$ (b) $5\times\frac{4}{3}\times\frac{9}{11}\times2\frac{3}{4}$.
3. Reduce to the simplest form.

$$\text{(a)}\ \frac{7\frac{1}{2}}{2\frac{1}{4}} \qquad \text{(b)}\ \frac{3\frac{1}{3}\times4\frac{3}{4}}{7\frac{1}{8}\times6\frac{1}{6}}$$

4. If $108\frac{3}{20}$ is the dividend and $5\frac{4}{5}$ is the divisor, what is the quotient?

5. If a cubic foot of water weighs $62\frac{1}{2}$ pounds and steel is $7\frac{4}{5}$ times as heavy as water, how much does a cubic foot of steel weigh?

6. The metal lining of a tank weighs $3\frac{3}{5}$ pounds to the square foot. How many pounds will be required to line a tank whose inside surface is $237\frac{1}{2}$ square feet?

7. A piece of copper wire weighs $\frac{9}{16}$ ounce per inch. How many inches in a roll of wire weighing $15\frac{21}{256}$ ounces?

8. A man owns $\frac{3}{5}$ of a building which is worth \$20,000. If he gives his son $\frac{1}{2}$ of his share, how much has he left?

9. John is 12 years old. James is $2\frac{1}{2}$ times as old as John. Albert is twice as old as James. How old is Albert?

10. If River A is $416\frac{13}{15}$ miles long and River B is only $\frac{3}{5}$ as long, what is the length of river B?

Suggestion. Check over all the examination problems and your solutions and answers again, to make sure you have used the right rules and have not made errors in calculations. Did you reduce all answers to simplest form and lowest terms?

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. (a) Multiply $\frac{3}{4}\times\frac{16}{3}\times\frac{18}{24}\times\frac{12}{9}$.

First do all the cancellation possible.

$$\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}}\times\frac{\overset{4}{\cancel{16}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{1}{\overset{\cancel{3}}{\cancel{18}}}}{\underset{1}{\underset{\cancel{4}}{\cancel{24}}}}\times\frac{\overset{1}{\overset{\cancel{3}}{\cancel{12}}}}{\underset{1}{\underset{\cancel{3}}{\cancel{9}}}}=4$$

(b) Multiply $\frac{7}{32}\times16\times3\frac{1}{4}$.

Here we have to state the 16 as a fraction and change $3\frac{1}{4}$ to an improper fraction. Then cancel as much as possible.

$$\frac{7}{\underset{8}{\cancel{32}}}\times\frac{\overset{1}{\overset{\cancel{4}}{\cancel{16}}}}{1}\times\frac{13}{\underset{1}{\cancel{4}}}=\frac{7\times1\times13}{8\times1\times1}=\frac{91}{8}=11\frac{3}{8}$$

2. (a) Divide $\frac{13}{10}$ by $6\frac{1}{4}$.

See **Rule 7.** Here we must change $6\frac{1}{4}$ to an improper fraction.

$$\frac{13}{10}\div\frac{25}{4}=\frac{13}{\underset{5}{\cancel{10}}}\times\frac{\overset{2}{\cancel{4}}}{25}=\frac{26}{125}$$

(b) Divide 39 by $\frac{3}{2}$. (See **Rule 8.**)

$$39\div\frac{3}{2}=\frac{39}{1}\div\frac{3}{2}=\frac{\overset{13}{\cancel{39}}}{1}\times\frac{2}{\underset{1}{\cancel{3}}}=26$$

3. Reduce to simplest form and lowest terms.

$$\frac{\frac{4}{\frac{1}{2}}\times\frac{1}{3}}{\left(\frac{1}{2}+\frac{4}{2}\right)\times\left(\frac{2}{3}-\frac{1}{3}\right)}$$

Solution

$$\overset{(a)}{\frac{\frac{4}{\frac{1}{2}}\times\frac{1}{3}}{\left(\frac{1}{2}+\frac{4}{2}\right)\times\left(\frac{2}{3}-\frac{1}{3}\right)}}=\overset{(b)}{\frac{8\times\frac{1}{3}}{\frac{5}{2}\times\frac{1}{3}}}=\overset{(c)}{\frac{\frac{8}{3}}{\frac{5}{6}}}=\overset{(d)}{3\frac{1}{5}}$$

Explanation of steps:

(*a*) The problem as given.

(*b*) Here we divided 4 by $\frac{1}{2}$ and obtained 8. (See **Rule 8.**)

$$4\div\frac{1}{2}=\frac{4}{1}\div\frac{1}{2}=\frac{4}{1}\times\frac{2}{1}=8$$

We added $\frac{1}{2}+\frac{4}{2}$ and got $\frac{5}{2}$.

We subtracted $\frac{1}{3}$ from $\frac{2}{3}$ and got $\frac{1}{3}$.

The parentheses were removed as the indicated operations within them were performed.

(*c*) See **Rule 1.**

$$8\times\frac{1}{3} \text{ or } \frac{1}{3}\times 8 \text{ mean the same.}$$

Then we can multiply $\frac{1}{3}$ by 8.

$$\frac{1}{3}\times 8=\frac{1\times 8}{3}=\frac{8}{3}=\text{numerator.}$$

This is put in Step *c*.

To multiply $\frac{5}{2}$ by $\frac{1}{3}$ use **Rule 2.**

$$\frac{5}{2}\times\frac{1}{3}=\frac{5\times 1}{2\times 3}=\frac{5}{6}=\text{denominator.}$$

(*d*) Here we have to divide $\frac{8}{3}$ by $\frac{5}{6}$. (Use **Rule 7.**)

$$\frac{8}{3}\div\frac{5}{6}=\frac{8}{\cancel{3}}\times\frac{\overset{2}{\cancel{6}}}{5}=\frac{16}{5}=3\frac{1}{5} \text{ Answer.}$$

4. Here we must remember what we learned in Section 1. Be sure you know what a minuend and a remainder are. To simplify this problem we will put these terms in terms of figures.

$46\frac{4}{5}$ (minuend)
? (subtrahend)
$26\frac{4}{5}$ (remainder)

The three terms are expressed in their proper position and we see that we must find the subtrahend. We can subtract the remainder from the minuend and find the subtrahend. Thus $46\frac{4}{5}-26\frac{4}{5}=20$.

Proof

$46\frac{4}{5}$ (minuend)
20 (subtrahend)
$26\frac{4}{5}$ (remainder)

5. If your father has lived $2\frac{1}{2}$ times longer than you have, then he has lived $2\frac{1}{2}$ times $18\frac{1}{3}$, or $2\frac{1}{2}\times 18\frac{1}{3}$. (See **Rule 4.**)

$$2\tfrac{1}{2}=\frac{5}{2}$$
$$18\tfrac{1}{3}=\frac{55}{3}$$

Then $\frac{5}{2}\times\frac{55}{3}=\frac{275}{6}=45\frac{5}{6}$ (Ans.)

6. If one foot of rail weighs $110\frac{1}{2}$ pounds then $\frac{1}{2}$ foot will weigh $\frac{1}{2}$ of $110\frac{1}{2}$. (See **Rule 2**). First change $110\frac{1}{2}$ to an improper fraction.

$$110\tfrac{1}{2}=\frac{221}{2}$$

Then $\frac{1}{2}\times\frac{221}{2}=\frac{221}{4}=55\frac{1}{4}$ pounds.

7. First add up the entire amount of cookies purchased.

$$2\tfrac{1}{3}+2+\tfrac{1}{3}=4\tfrac{2}{3} \text{ pounds}$$

Then to find the cost we would multiply $4\frac{2}{3}$ by $\frac{1}{3}$. (See **Rule 2.**)

$$4\tfrac{2}{3}=\frac{14}{3}$$

Then $\frac{14}{3}\times\frac{1}{3}=\frac{14}{9}=1\frac{5}{9}$.

So all the cookies cost $1\frac{5}{9}$ dollars.

8. This problem should be expressed in terms of a complex fraction.

$$\frac{2\frac{6}{10}\times 3}{}$$

Solution

$$\overset{(a)}{\frac{2\frac{6}{10}\times 3}{\frac{2}{3}\times\frac{6}{8}}} = \overset{(b)}{\frac{7\frac{4}{5}}{\frac{2}{3}\times\frac{6}{8}}} = \overset{(c)}{\frac{7\frac{4}{5}}{\frac{1}{2}}} = \overset{(d)}{\frac{\frac{39}{5}}{\frac{1}{2}}} = \overset{(e)}{\frac{39}{5}\times\frac{2}{1}} = \overset{(f)}{\frac{78}{5} = 15\frac{3}{5}}$$

Explanation of steps:

(*a*) Problem in terms of a complex fraction.

(*b*) Following Rule 3, we first multiply $\frac{6}{10}\times 3 = \frac{6\times 3}{10} = \frac{18}{10}$ or $1\frac{4}{5}$. Then we multiply $2\times 3 = 6$. Next we add the two products, $6+1\frac{4}{5} = 7\frac{4}{5}$ = numerator.

(*c*)
$$\frac{2}{3}\times\frac{6}{8} = \frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{\overset{1}{\cancel{3}}}{\cancel{6}}}{\underset{\cancel{4}}{\cancel{8}}} = \frac{1}{2} = \text{denominator}$$

(*d*) Change $7\frac{4}{5}$ to an improper fraction.

(*e*) $\frac{\frac{39}{5}}{\frac{1}{2}} = \frac{39}{5}\div\frac{1}{2} = \frac{39}{5}\times\frac{2}{1}$

(*f*) The answer.

9. We can solve this problem in three steps.

$$\frac{2}{3}\text{ of }24 = \frac{2}{3}\times 24 = \frac{2\times\cancel{24}}{\cancel{3}} = \frac{16}{1} = 16 = \text{numerator}$$

$$2\frac{1}{8}\times\frac{3}{2} = \frac{17}{8}\times\frac{3}{2} = \frac{51}{16} = \text{denominator.}$$

Then $\frac{16}{\frac{51}{16}} = 16\div\frac{51}{16} = \frac{16}{1}\div\frac{51}{16} = \frac{16}{1}\times\frac{16}{51} = \frac{256}{51} = 5\frac{1}{51}$ (Ans.)

10. Here we can solve the problem in three steps.

$$\frac{\frac{6}{10}\times\frac{24}{18}\times\frac{3}{36}\times\frac{48}{24}}{\frac{\frac{1}{3}}{2\frac{3}{5}}}\ \text{the problem stated.}$$

$$\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{6}}}{\underset{5}{\cancel{10}}}\times\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{24}}}{\underset{\underset{3}{\cancel{9}}}{\cancel{18}}}\times\frac{\overset{1}{\cancel{3}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{36}}}\times\frac{\overset{2}{\cancel{48}}}{\underset{1}{\cancel{24}}} = \frac{2}{15} = \text{numerator}$$

$$\frac{\frac{1}{3}}{2\frac{3}{5}} = \frac{\frac{1}{3}}{\frac{13}{5}} = \frac{1}{3}\div\frac{13}{5} = \frac{1}{3}\times\frac{5}{13} = \frac{5}{39} = \text{denominator.}$$

Then $\frac{2}{15}\div\frac{5}{39} = \frac{2}{\cancel{15}}\times\frac{\overset{13}{\cancel{39}}}{5} = \frac{26}{25} = 1\frac{1}{25}$ Answer

PRACTICAL MATHEMATICS

Section 5

Lesson 1

For Step 1, recall what a fraction is, because a decimal is a fraction with the denominator 10; 100; 1000, etc. The majority of the civilized countries use the metric system in figuring distances and many other dimensions in place of our English system of feet, yards, quarts, etc., and the metric system is based on decimal units. For Step 2, learn what a decimal is and how to read and write decimals. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

DECIMALS

In Section 1, you learned that the decimal system of notation is used in writing integers (whole numbers). For instance, take the number 6666666. This number is really the sum of

6.	six
60.	sixty
600.	six hundred
6000.	six thousand
60000.	sixty thousand
600000.	six hundred thousand
6000000.	six million
6666666.	six million, six hundred sixty-six thousand, six hundred sixty-six

Notice that these names do not have "th" at the end.

Thus you see that the first number at the right in the sum is really a number 6, and naturally it is a whole number between 0 and 10. In other words, it is in the units place (unit means one). The number has only one figure in it. The number composed of the first two figures from the right is 66. It is a whole number between 10 and 100. In other words the second figure from the right in the sum is in the tens place. The number composed of the first three figures from the right is 666. It is a number between 100 and 1000. In other words, the third figure in the sum is in the hundreds place. This reasoning is applied to the rest of the figures in the sum.

From this you see that a figure in the tens place is just ten times the one in the units place, and the one in the hundreds place is ten times the one in the tens place, and so on. Therefore if you move a figure to the left, place by place, you increase its value ten times for each place that it moves. Now if instead of moving the figure to the left of the point, place by place, you move it to the right of the point, place by place, you decrease its value ten times for each place. Take the same number .6666666 but with the point at the left end as shown. It is made up of the sum of the following quantities:

.0000006	six ten millionths
.000006	six millionths
.00006	six hundred thousandths
.0006	six ten thousandths
.006	six thousandths
.06	six hundredths
.6	six tenths
.6666666	

Notice that these names have "ths" at the end.

Thus you see that by placing ciphers between the point and the number, the number keeps moving farther away to the right from the point, and that for each place that it moves to the right, it becomes smaller and is one-tenth of the value that it had before. The six just to the right of the point is one-tenth as large as when it was just to the left of the point. The same reasoning applies to the six as it moves farther to the right. When it is two places from the point, it is one-hundredth of what it was on the left of the point. When it is three places to the right of the point, it is one-thousandth of what it was on the left of the point, and so on as it keeps moving to the right.

Any fraction having 10, 100, 1000, or other multiple of ten, for the denominator may be written without the denominator (converted into its decimal equivalent) by placing, before the numerator, as many ciphers as may be necessary to make **as many figures** in the numerator as there were **ciphers** in the denominator, then placing a period in front. This period is called a decimal point, and it indicates what the denominator is without writing it. Thus $\frac{7}{10}=.7$; $\frac{7}{100}=.07$; $\frac{7}{1000}=.007$; $\frac{77}{100}=.77$; $\frac{77}{1000}=.077$; $\frac{777}{1000}=.777$; etc.

Table I gives the names of the places on each side of the decimal point so that you may know what to call the places when you read them.

TABLE I

Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units	Decimal Point	Tenths	Hundredths	Thousandths	Ten Thousandths	Hundred Thousandths	Millionths	Ten Millionths	Hundred Millionths	Billionths
7,	6	5	4,	3	2	1	.	1	2	3	4	5	6	7	8	9

Notice that in order to make the table clearer, the figures also indicate the number of places to the right or to the left of the decimal point.

When there is no whole number as in .5, .05, etc., the quantity is called a **pure decimal.**

When a whole number and a decimal are written together, the number is called a **mixed decimal.** Thus 3.75 is a mixed decimal, and is read 3 and 75 hundredths. Notice carefully that the word **and** is read at the decimal point. Shop workers and others who use decimals and mixed decimals a great deal often say **point** at the decimal point. Thus 2.5 would be read two point five.

The United States money system is a very good illustration of decimals. The dollar is the basis of the system, and it stands for 100 units or cents, and each unit or cent is one hundredth of a dollar or $.01. Therefore, if you had 25 cents you would have 25 hundredths of a dollar, or $.25.

A dime is worth 10 hundredths of a dollar, or $.10, therefore, if you had two dimes, you would have twenty hundredths of a dollar. This would be written $.20.

Reading Decimals. After studying Table I carefully so that you know the names of the decimal places and studying the principles on writing decimals, you will find it quite easy to read decimals. In order to read a decimal, read the number as though it were a whole number, or integer, and then call the name of the decimal place of the last right-hand figure according to Table I.

ILLUSTRATIVE EXAMPLES

1. Read the decimal .68

The last right-hand figure (8) occupies the place of the hundredths in Table I, so we add that name to the reading of the number. Then .68 is read sixty-eight hundredths.

2. Read the decimal .0567

The last figure to the right (7) occupies the place of the ten thousandths in Table I. So, .0567 is read five-hundred sixty-seven ten thousandths.

3. Read the number 72.093

The whole number is read by itself. In reading the decimal part, we see that the last figure (3) occupies the thousandths place in Table I, so the number 72.093 is read seventy-two and ninety-three thousandths.

If you would convert the decimal quantities of the problems into fractions, you would have for each denominator a figure 1 with a number of ciphers following it equal to the number of figures in the numerator. For instance, .71 would be $\frac{71}{100}$. Then if you would call the fraction aloud, you would hear the same name as given to the corresponding decimal.

PRACTICE PROBLEMS

Read the following numbers:

1.	.7	Ans. Seven-tenths
2.	.0091	Ans. Ninety-one ten thousandths
3.	6.31	Ans. Six and thirty-one hundredths
4.	25.0002	Ans. Twenty-five and two ten-thousandths
5.	3.000003	Ans. Three and three millionths

We have referred you to Table I as an aid in reading decimals, but you should learn the names of each succeeding decimal place so that you may read decimal quantities without the aid of the table. In reading a decimal, begin at the number or zero to the right of the decimal point and call it tenths, then move to the next place to the right and call it hundredths, and so on until you read the last number to the right. This last figure to the right gives you the place by which the decimal numbers will be called.

For example, read the decimal quantity .00054. Commencing at the first 0 to the right of the decimal point, we call it tenths, the second 0 is hundredths, the third 0 is thousandths, the 5 is ten thousandths, the 4 is hundred thousandths; then the quantity is fifty-four hundred-thousandths. Go over the Practice Problems again and use this method instead of looking at the Table.

Writing Decimals. In writing a decimal number, write the number as though it were a whole number and then place the decimal point so that the last figure to the right is in the place named by the decimal number.

ILLUSTRATIVE EXAMPLES

1. Write thirteen hundredths as a decimal.

The last figure of 13 must be in the hundredths place. We have learned that the second place to the right of the decimal point is the hundredths place, so thirteen hundredths is written .13.

2. Write twenty-six thousandths as a decimal.

Write 26 and then place the decimal point so that the figure 6 is in the thousandths place. This is the third place. One cipher must be placed in front of 26 so that 6 will be in the third place. So twenty-six thousandths is written .026. This decimal point also indicates that if this decimal were written as a fraction, the denominator would have 3 ciphers, thus $\frac{26}{1000}$.

3. Write two hundred seven thousand, eighty-three ten millionths as a decimal.

Write the entire number 207083, then place the point so that the last figure (3) is in ten millionths place. This is the seventh place to the right of the decimal point. It is necessary to place a cipher before our number in order to get a seventh place. The required decimal is .0207083.

4. Write in figures: Two hundred and seven thousandths.

Note that the word **and** after the two hundred indicates that 200 is a whole number. Seven thousandths is the decimal part. The figure 7 must be in the thousandths or third place to the right of the decimal point. It is necessary, then, to place two ciphers before it. The required number is 200.007.

PRACTICE PROBLEMS

Write the following in figures:

1. Four and seven tenths	Ans.	4.7
2. Seventy-five hundredths	Ans.	.75
3. One hundred and forty-four thousandths	Ans.	100.044
4. One millionth	Ans.	.000001
5. Four and eight thousandths	Ans.	4.008

When a cipher is added to the right of a whole number, it is the same as multiplying the whole number by 10. This is not true of decimals. Putting a cipher, or ciphers, at the right of a decimal number does not change its value. Thus, .53 is of exactly the same value as .5300

A decimal quantity in which figures repeat themselves indefinitely, such as .454545, etc., is called a **recurring, repeating, or circulating decimal,** and the group of figures which repeat are called the **repetend.**

A decimal may also contain a fraction. For instance $.56\frac{1}{3}$. Such a decimal if written in fractional form becomes a complex fraction, as $\frac{56\frac{1}{3}}{100}$. This is called a **complex decimal.**

Lesson 2

For Step 1, remember that since the common fractions and the decimals are used in figuring, many occasions will arise where quantities in either system will have to be changed to equivalent values in the other system. For Step 2, learn how to change decimals to fractions. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Changing Decimals to Common Fractions

To change a decimal to a common fraction, use the figures in the quantity as a numerator and place for the denominator the figure **one** followed by as many ciphers as there are figures to the right of the decimal point in the quantity.

ILLUSTRATIVE EXAMPLES

1. Change the decimal .3 to a common fraction.

We place the 3 as the numerator of the fraction and place 1 followed by one cipher as the denominator. $.3=\frac{3}{10}$

2. Change the decimal .045 to a common fraction.

We place 045 as the numerator, and 1 followed by three ciphers as the denominator, since there are three figures in the number. $.045=\frac{045}{1000}$ or $\frac{45}{1000}$

3. Change the decimal 7.002 to a common fraction.

The numerator of the fraction will be 7002, and the denominator will be 1 followed by three ciphers, since there are three figures to the right of the decimal point. The fraction is, then, $\frac{7002}{1000}$

PRACTICE PROBLEMS

Change the following decimals to common fractions:

1. .325 Ans. $\frac{325}{1000}$
2. .004 Ans. $\frac{4}{1000}$
3. .0205 Ans. $\frac{205}{10000}$
4. 9.3 Ans. $\frac{93}{10}$

Lesson 3

For Step 1, recall addition and subtraction of integral numbers. For Step 2, learn the method of adding and subtracting decimals. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Adding and Subtracting Decimals

You learned in Section 1 how to add and subtract whole numbers. You also learned that only similar quantities may be combined in addition and subtraction. This same fact must be applied in adding and subtracting decimals.

Rule. *In order to add or subtract pure or mixed decimals, write the numbers so that the decimal points are under each other. Thus units will be added to units, tens to tens, hundredths to hundredths, etc. Then add or subtract as in whole numbers. When that is done, place the decimal point of the result directly below the other decimal points.*

ILLUSTRATIVE EXAMPLES

1. Add 27.072 and 8.923

Place by rule	27.072
Add	8.923
Point off by rule	35.995

2. Add 100.207 to 20.003

Place by rule	100.207
Add	20.003
Point off by rule	120.210

3. Add 0.0004, 24.345, 740.01, and 1.2345

Place by rule	0.0004
	24.345
	740.01
Add	1.2345
Point off by rule	765.5899

4. Subtract 8.0003 from 21.43

Place by rule	21.4300
Subtract	8.0003
Point off by rule	13.4297

5. Subtract 275.0005 from 3000.024

Place by rule	3000.0240
Subtract	275.0005
Point off	2725.0235

6. Subtract .100203 from .30405

Place by rule	.304050
Subtract	.100203
Point off	.203847

7. Subtract eighty-nine cents from one dollar and sixty-four cents.

Place by rule	\$1.64
Subtract	.89
Point off	\$0.75 or seventy-five cents

NOTE. If the numbers are complex, they will have to be simplified or reduced unless the fractional parts can be combined easily.

8. Add $.427\frac{1}{3}$ to $.701\frac{2}{3}$

These can easily be added for the fractions combine to equal a unit. $\frac{1}{3}+\frac{2}{3}=\frac{3}{3}$ or 1

Thus	$.427\frac{1}{3}$
Add	$.701\frac{2}{3}$
Point off	1.129

9. Subtract $27.083\frac{2}{7}$ from $120.127\frac{5}{7}$

Place by rule	$120.127\frac{5}{7}$
Subtract	$27.083\frac{2}{7}$
Point off	$93.044\frac{3}{7}$

10. Add $18.24\frac{3}{8}$ to $117.05\frac{1}{4}$

Add the fractions $\frac{3}{8}+\frac{1}{4}=\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$

Place by rule	$18.24\frac{3}{8}$
Add	$117.05\frac{2}{8}$
Point off	$135.29\frac{5}{8}$

11. Subtract $127.58\frac{1}{4}$ from $139.327\frac{5}{8}$

In this case the fractions are not in the same vertical place. To get them in position one under the other, we can change the denominator of the $\frac{1}{4}$. The $\frac{1}{4}$ is in the **hundredths** place or it is $\frac{\frac{1}{4}}{100}$. To get it directly under the other fraction, we must get it in the thousandths place, that is, the denominator must be multiplied by 10. In order that the value be not changed, multiply the numerator by 10 also. $\frac{\frac{1}{4}}{100}=\frac{\frac{1}{4}\times 10}{100\times 10}=\frac{2\frac{1}{2}}{1000}$, so we can annex $2\frac{1}{2}$ to .58, giving $127.582\frac{1}{2}$

Place by rule	$139.327\frac{5}{8}$
Subtract	$127.582\frac{1}{2}$
Point off	$11.745\frac{1}{8}$ $(\frac{5}{8}-\frac{1}{2}=\frac{1}{8})$

PRACTICE PROBLEMS

1. Three pieces of land have acreages as follows: 21.5, 18.66, and 29.325. What is the total? Ans. 69.485 acres.

2. A barrel holding 31.5 gallons had 8.5 gallons and 7.25 gallons sold. How much was left? Ans. 15.75 gallons.

3. A boy has an allowance of \$12.50 per month. He spent \$3.2$\frac{1}{2}$ for skates, \$1.75 for a cap, \$3.15 for lunches, and 15¢ for candy. How much had he left? Ans. \$4.20

4. A cistern had 64.5 barrels of water. 29.25$\frac{1}{3}$ and 15.6$\frac{3}{8}$ barrels were pumped out. How much was left? Ans. 19.60$\frac{11}{12}$ bbs.

5. An iron casting weighed in the rough 22.75 pounds, and when finished, it weighed only 16.875 pounds. How much had been taken off in the process? Ans. 5.875 pounds.

Lesson 4

For Step 1, recall multiplication of integral numbers. For Step 2, learn the method of multiplying decimals. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Multiplication of Decimals

In Section 4 you learned that when you multiply **a fraction** by an integer the denominator does not change **unless you use** cancellation.

Thus $\frac{3}{10} \times 3 = \frac{9}{10}$. Now, if we reduce $\frac{3}{10}$ to a decimal we get .3. So that multiplying .3 by 3 will give the same value as multiplying $\frac{3}{10}$ by 3. But multiplying $\frac{3}{10}$ by 3 gives $\frac{9}{10}$ and $\frac{9}{10}$ reduced to a decimal is .9. Therefore .3 multiplied by $3 = .9$.

When you multiply two fractions together, the conditions are altered. In this case the denominators as well as the numerators are multiplied, thus giving a different resulting denominator.

For example, take $\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$

Now reduce these to decimals and you get $.3 \times .3 = .09$

From these examples the following rule is deduced:

Rule. *To multiply decimals, multiply as in whole numbers. Then begin at the right of the product and point off toward the left a number of decimal places that will equal the sum of the number of decimal places in the quantities multiplied. Prefix ciphers when necessary.*

ILLUSTRATIVE EXAMPLES

1. Multiply .875 by .37

Solution

Instruction

Multiply as in whole numbers. There are 3 decimal places in multiplicand and 2 decimal places in multiplier, so there must be 3+2 or 5 decimal places in the product. Point off five decimal places from right.

Operation

$$
\begin{array}{r}
.875 \\
\underline{.37} \\
6125 \\
\underline{2625} \\
.32375
\end{array}
$$

2. Multiply 4.7023 by 1.092

Solution

Instruction

Multiply as in whole numbers. There are 4 decimal places in multiplicand and 3 decimal places in multiplier so there must be 4+3 or 7 decimal places in the product. Point off seven decimal places from right.

Operation

$$
\begin{array}{r}
4.7023 \\
\underline{1.092} \\
94046 \\
423207 \\
\underline{470230} \\
5.1349116
\end{array}
$$

3. Multiply .00123 by .01023

Solution

Instruction

Multiply as in whole numbers. There must be 5+5 or 10 decimal places in the product. To get these ten places we must annex 4 ciphers to the left of the result obtained after multiplying.

Operation

$$
\begin{array}{r}
.00123 \\
\underline{.01023} \\
369 \\
246 \\
\underline{1230} \\
.0000125829
\end{array}
$$

4. Multiply $802.03\frac{1}{3}$ by .9006

Solution

Instruction	*Operation*
Multiply fraction separately and add to product of other numbers, disregarding the decimal points. There must be 2+4 or 6 decimal places in the product.	$.9006 \times \frac{1}{3} = .3002$ 802.03 .9006 481218 72182700 3002 722.311220

5. Multiply $3.257\frac{1}{7}$ by $8.13\frac{1}{4}$

Solution

Instruction	*Operation*
Step 1 Reduce $\frac{1}{4}$ to a decimal Leave $\frac{1}{7}$ to be handled as in preceding problems.	**Step 1** $\frac{1}{4} = .25$
Step 2 Annex 25 to rest of decimal and then multiply $\frac{1}{7}$ by the complete number.	**Step 2** $8.13\frac{1}{4} = 8.1325$ $8.1325 \times \frac{1}{7} = 1.1618-$
Step 3 Multiply the mixed decimals and add on product of **Step 2**, disregarding the decimal point. 3+4=7 decimal places to point off in the product.	**Step 3** 3.257 8.1325 16285 6514 9771 3257 26056 11618− 26.4887143−

PRACTICE PROBLEMS

1. $.35 \times 4$ Ans. 1.40 or 1.4
2. $.785 \times 25$ Ans. 19.625

3.	$.287 \times .356$	Ans. .102172
4.	$.002 \times .014$	Ans. .000028
5.	1.0034×2.503	Ans. 2.5115102
6.	$0.0333\frac{1}{3} \times 9$	Ans. .3000 or .3
7.	$14.285\frac{5}{7} \times 7$	Ans. 100.000 or 100.
8.	32.25×2.376	Ans. 76.62600 or 76.626
9.	$.2717\frac{1}{7} \times 7.07$	Ans. 1.92102

Short Methods of Multiplication. There are some short methods that can be used to advantage in multiplication of decimals.

ILLUSTRATIVE EXAMPLES

1. To multiply a decimal or a mixed decimal by 10, 100, 1000, etc., it is only necessary to move the decimal point as many places to the right as there are ciphers in the multiplier. If there are not figures enough, annex ciphers to the right.

$.0076 \times 10 = 0.076$
$.0076 \times 100 = 00.76$ or $.76$
$.0076 \times 1000 = 007.6$ or 7.6
$.0076 \times 10000 = 0076.$ or $76.$
$.0076 \times 100000 = 00760.$ or $760.$ (One cipher annexed as there are five ciphers in the multiplier.)

2. To multiply by 50, multiply by 100 and divide by 2, since $50 = 100 \div 2$

Multiply 6.42 by 50
$6.42 \times 100 = 642.$

(According to example 1, we multiply by 100 by moving the decimal point 2 places to the right.)

$642. \div 2 = 321.$

3. To multiply by 25, multiply by 100 and divide by 4, since $25 = 100 \div 4$

Multiply 7.2 by 25
$7.2 \times 100 = 720.$
$720 \div 4 = 180$

4. To multiply by $12\frac{1}{2}$, multiply by 100 and divide by 8, since $12\frac{1}{2} = 100 \div 8$

Multiply 8.56 by $12\frac{1}{2}$
$8.56 \times 100 = 856$
$856 \div 8 = 107$

5. To multiply by $33\frac{1}{3}$, multiply by 100 and divide by 3, since $33\frac{1}{3} = 100 \div 3$

Multiply 0.72 by $33\frac{1}{3}$
$0.72 \times 100 = 72$
$72 \div 3 = 24$

6. To multiply by 125, multiply by 1000 and divide by 8, since $125 = 1000 \div 8$

Multiply 23.8 by 125
$23.8 \times 1000 = 23800$
$23800 \div 8 = 2975$

PRACTICE PROBLEMS

Solve the following by the short methods so as to get familiar with them.

1. 75.22×50 Ans. 3761
2. $48.80 \times 12\frac{1}{2}$ Ans. 610
3. 28.24×125 Ans. 3530
4. $48.3 \times 33\frac{1}{3}$ Ans. 1610
5. 75.2×25 Ans. 1880

Ordinarily when accuracy to a certain decimal place is desired, the figures beyond that place are dropped. If the first figure to be dropped is 5 or more, add one to the preceding figure to the left; and if less than 5, it is dropped without adding anything. For instance, $.027 \times .748$ to 5th decimal place.

$$.027 \times .748 = .020196$$

The sixth place is 6, so add one to 9, giving 10, and drop 6
This gives .0202, which is reasonably accurate
Find product of $.34 \times .56$ to second decimal place

$$.34 \times .56 = .1904$$

As the last figure is four, it can be dropped giving product of .19 which is accurate that far.

Lesson 5

For Step 1, recall division of integral numbers. For Step 2, learn the method of dividing decimals. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Division of Decimals

In beginning the study of division of decimals it is well to consider what is done in the division of common fractions and make the comparison using decimal equivalents as was shown in multiplication.

$\frac{6}{10} \div 2 = \frac{3}{10}$

Reducing to decimals you get $.6 \div 2 = .3$

$$\text{Also } \frac{6}{10} \div \frac{2}{10} = \frac{\overset{3}{\cancel{6}}}{\cancel{10}} \times \frac{\cancel{10}}{\cancel{2}} = 3$$

Using the same fractions reduced to decimal equivalents, you get $.6 \div .2 = 3.$

You learned in Section 4 that when you divide any number by a proper fraction the result is greater than the number divided. The same rule holds good in decimals.

Using a whole number for the dividend, we get $6 \div \frac{2}{10} = \overset{3}{\cancel{6}} \times \frac{10}{\cancel{2}} = 30$

Using decimal equivalents, $6 \div .2 = 30$

The above results summarized give the following:

$.6 \div 2 = .3$	1 decimal place in dividend and no decimal place in divisor gives 1 decimal place in quotient.
$.6 \div .2 = 3.$	1 decimal place in dividend and 1 decimal place in divisor gives integers in quotient.
$6. \div .2 = 30.$	No decimal place in dividend and 1 in divisor gives 2 places to left of point in quotient.

There is thus a relation between the number of decimal places in the dividend, in the divisor, and in the quotient.

Rule. *Divide as though dividend and divisor were whole numbers. Then point off from right toward left as many decimal places as the difference between the number of decimal places in the dividend and in the divisor.*

If the number of decimal places in the dividend is less than the number of decimal places in divisor, annex ciphers to the dividend, remembering that there must be at least as many decimal places in dividend as in divisor.

NOTE. There may be as many ciphers annexed as desired, as you have learned that adding ciphers to right of a decimal does not alter its value.

ILLUSTRATIVE EXAMPLES

1. Divide 37.24588 by 124.

Solution

Instruction	*Operation*
Step 1 Divide as with whole numbers disregarding the decimals.	**Step 1** $$\begin{array}{r} .30037 \\ 124\overline{)37.24588} \\ \underline{37\ 2} \\ 0458 \\ 372 \\ 868 \\ 868 \end{array}$$
Step 2 There are 5 decimal places in the dividend. There are no decimal places in the divisor. The difference between 5 and 0 is 5. So there must be 5 decimal places in the quotient. Count 5 places from the right toward the left and place the decimal point.	**Step 2** .30037 Ans.

2. Divide .876447 by 27.

Solution

Instruction	*Operation*
Step 1 Divide the numbers disregarding the decimals entirely.	**Steps 1 and 2**
Step 2 There are 6 decimal places in the dividend and 0 decimal places in the divisor. The difference between 6 and 0 is 6. So there must be 6 decimal places in the quotient. Place the decimal point by counting 6 places from the right toward the left in quotient. Place the decimal point. Note that we had to prefix a cipher in order to have 6 decimal places.	$$\begin{array}{r} .032461 \\ 27\overline{).876447} \\ \underline{81} \\ 66 \\ \underline{54} \\ 124 \\ \underline{108} \\ 164 \\ \underline{162} \\ 27 \\ 27 \end{array}$$ .032461 Ans.

3. Divide .231762 by .038

Solution

Instruction	*Operation*
Step 1 Divide as though both quantities were whole numbers. Since 0 has no value at the first of whole numbers we can disregard it in the divisor.	**Step 1** $$\begin{array}{r} 6.099 \\ .038\overline{).231762} \\ \underline{228} \\ 376 \\ \underline{342} \\ 342 \\ 342 \end{array}$$
Step 2 There are 6 decimal places in dividend and 3 in divisor. The difference between 6 and 3 is 3. So there must be 3 decimal places in the quotient. Count back from the right of quotient 3 places. Place the decimal point.	**Step 2** 6.099 Ans.

4. Divide 28.65 by 31.4326

Solution

Instruction

Step 1

Notice that there are more decimal places in the divisor than there are in the dividend. (Refer to second part of the rule.) We must annex ciphers to the dividend. We may annex any number of ciphers remembering that there must be at least as many decimal places in the dividend as in the divisor. Then divide as in whole numbers.

Operation

Step 1

$$\begin{array}{r} .91 \\ 31.4326\overline{)28.650000} \\ 2828934 \\ \hline 360660 \\ 314326 \\ \hline 46334 \end{array}$$

Step 2

After annexing the ciphers our dividend is 28.650000, having 6 decimal places. The divisor has 4 decimal places. Difference is 2. So there must be 2 decimal places in the quotient. Place the decimal point. (Since there is a remainder, we may write the quotient .91+)

Step 2

.91+ Ans.

5. Divide 7 by 13.

Solution

Instruction

Step 1

Expressed in decimal form. The problem is 7.÷13., the decimal points being at the right of each number. There are more decimal places in the divisor than in the dividend so we annex ciphers to the dividend and proceed with the division.

Operation

Step 1

$$\begin{array}{r} .538 \\ 13\overline{)7.000} \\ 6\,5 \\ \hline 50 \\ 39 \\ \hline 110 \\ 104 \\ \hline 6 \end{array}$$

Step 2
After annexing the ciphers, our dividend is 7.000 in which there are 3 decimal places. There are 0 decimal places in the divisor, so there must be $3-0$ or 3 decimal places in the quotient. Place the decimal point.

Step 2

.538+ Ans.

6. Divide 1.5798 by 3.6645

Solution

Instruction

Step 1
If we try to divide these quantities as whole numbers, we find that the dividend is not divisible by the divisor even though there are as many decimal places in the dividend as in the divisor. Remembering that it is always permissible to annex as many ciphers as we wish to the right of a decimal number, annex several ciphers to the dividend and then divide.

Operation

Step 1

```
3.6645)1.5798

             .4311
3.6645)1.57980000
         1 46580
           114000
           109935
             40650
             36645
              40050
              36645
               3405
```

Step 2
After annexing the ciphers, our dividend was 1.57980000, which contains 8 decimal places. The divisor has 4 decimal places, so the quotient has $8-4$, or 4 decimal places. Place the decimal point. (Since there is a remainder, we may write the result .4311+)

Step 2

.4311+ Ans.

Changing Fractions to Decimal Numbers. The principles of division of decimals can be applied in changing fractions to decimal numbers because in a fraction, the numerator is a dividend and the denominator is a divisor. Therefore, to put a common fraction in the form of a decimal number divide the numerator by the denominator, following the procedure for division of decimals.

The following table will show you the equivalent decimals for some of the simpler fractions.

$\frac{1}{2}=.5$	$\frac{2}{4}=.5$	$\frac{5}{8}=.625$	$\frac{4}{5}=.8$
$\frac{1}{4}=.25$	$\frac{3}{8}=.375$	$\frac{7}{8}=.875$	$\frac{2}{5}=.4$
$\frac{1}{8}=.125$	$\frac{2}{3}=.66\frac{2}{3}$	$\frac{3}{5}=.6$	
$\frac{1}{3}=.33\frac{1}{3}$	$\frac{3}{4}=.75$	$\frac{1}{5}=.2$	

PRACTICE PROBLEMS

Perform the following divisions.

1. $3.036 \div .06$ Ans. 50.6
2. $3.728 \div .16$ Ans. 23.3
3. $.864 \div .024$ Ans. 36.
4. $10.044 \div .36$ Ans. 27.9
5. $12 \div .7854$ Ans. 15.27+
6. $2.34 \div .211$ Ans. 11.09+

$.125 \div 8000$ Ans. .0000156+

Change the following fractions to decimal numbers:

(*a*) $\frac{5}{31}$ Ans. .161+
(*b*) $\frac{71}{206}$ Ans. .344+
(*c*) $\frac{47}{150}$ Ans. .313+

Short Methods of Division. In division of decimals, as in multiplication, some short methods can be used to advantage.

ILLUSTRATIVE EXAMPLES

1. To divide by 10, 100, 1000, etc., move the decimal point one place to the left for each cipher in the divisor.

$19.536 \div 10 = 1.9536$ $19.536 \div 1000 = .019536$
$19.536 \div 100 = .19536$ $19.536 \div 10000 = .0019536$

NOTE. Ciphers must be prefixed as shown, when necessary. This of course is just the opposite of multiplication.

2. To divide by 50 multiply by 2 and move the decimal point 2 places to the left because $50 = 100 \div 2$.

Divide 82.054 by 50
$82.054 \times 2 = 164.108$
Move decimal point 2 places to left $= 1.64108$

3. To divide by 25 multiply by 4 and move decimal point 2 places to the left as $25 = 100 \div 4$.

Divide 75.032 by 25
$75.032 \times 4 = 300.128$
Move decimal point 2 places to left $= 3.00128$

4. To divide by $33\frac{1}{3}$ multiply by 3 and move decimal point 2 places to the left, as $33\frac{1}{3} = 100 \div 3$.

$2.754 \times 3 = 8.262$
Move decimal point 2 places to left $= .08262$ (cipher is prefixed)

5. To divide by $16\frac{2}{3}$ multiply by 6 and move decimal point 2 places to left, as $16\frac{2}{3} = 100 \div 6$.

Divide .8275 by $16\frac{2}{3}$
$.8275 \times 6 = 4.9650$
Move decimal point 2 places to left $= .049650$

6. To divide by $12\frac{1}{2}$ multiply by 8 and move decimal point 2 places to left, as $12\frac{1}{2} = 100 \div 8$.

Divide .06523 by $12\frac{1}{2}$
$.06523 \times 8 = .52184$
Move decimal point 2 places to the left $= .0052184$ (2 ciphers must be prefixed)

7. To divide by 125 multiply by 8 and set the decimal point 3 places to the left, as $125 = 1000 \div 8$.

Divide 82.53 by 125
$82.53 \times 8 = 660.24$
Move decimal point 3 places to the left $= .66024$

8. To divide by $166\frac{2}{3}$ multiply by 6 and set the decimal point 3 places to the left, as $166\frac{2}{3} = 1000 \div 6$.

Divide 93.74 by $166\frac{2}{3}$
$93.74 \times 6 = 562.44$
Move decimal point 3 places to the left $= .56244$

Lesson 6

For Step 1, recall how to change a decimal to an equivalent fraction. In this lesson a decimal is changed to an equivalent fraction that must have a certain denominator. For Step 2, learn the method of changing decimals to equivalent fractions with a given denominator. For Step 3, work the Illustrative Example. For Step 4, work the Practice Problems.

Changing Decimals to Common Fractions with a Given Denominator

Sometimes it is necessary to change a decimal to a common fraction having a given denominator. The resulting fraction may not be exact, but it may be found nearly enough to the true value for practical purposes. A method is shown.

ILLUSTRATIVE EXAMPLE

1. Change .564 to the nearest 64th.

$$.564=\frac{564}{1000}$$

Multiply both terms of the fraction by 64

$$\frac{564\times 64}{1000\times 64}=\frac{36096}{64000}$$

Divide both terms of the fraction by 1000

$$\frac{36096}{64000}=\frac{36}{64}\text{ nearly. (The difference is very small)}$$

Rule. *To change a decimal to the nearest equivalent fraction having a desired denominator, multiply the decimal by the desired denominator. The result will be the numerator of the desired fraction.*

PRACTICE PROBLEMS

1. Change .756 to the nearest 12th Ans. $\frac{9}{12}$
2. Change .875 to the nearest 64th Ans. $\frac{56}{64}$
3. Change .719 to the nearest 32nd Ans. $\frac{23}{32}$

Lesson 7

For Step 1, recall the various operations you have learned in your study of decimal numbers. For Step 2, study the suggestions for solving written problems. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Problems Involving Decimals

The following suggestions are given as a help in solving problems in decimals.

1. Read the problem carefully to make sure that you understand what it says and what it asks for.
2. See how the information given is related to what you are required to find, and decide what process you must use.
3. Do not simply work with figures and numbers, trying one operation and then another till you arrive at the right answer. You will realize that such a method is of no use unless the answer is known. Avoid such a practice.
4. Reason out the problem using the information given in the problem and put down all statements in logical order. (In making these statements, it is helpful to remember that the number involving the quantity to be found should be put at the end of the line.) If the statements are correct the answer will come of itself.
5. Do your work carefully and neatly. Check each step to assure accuracy.
6. Prove your results, by checking, or by reversing the processes.

ILLUSTRATIVE EXAMPLES

1. If a train averages 40.5 miles per hour, how long will it take to travel 372.6 miles?

Solution

Instruction	*Operation*
The problem asks "How long?" So we are required to find the number of hours. Our statement, then, is:	$40.5)372.6(9.2$
	3645
	810
It travels 40.5 miles in 1 hour.	810
It travels 372.6 miles in 372÷40.5 hours = 9.2 hours.	

2. If an athlete makes the 100-yard dash in 9.6 seconds, how many yards did he travel in 1 second?

Solution

Instruction	*Operation*
We are required to find the number of yards. So in making your statement put the number involving yards at end of statement thus: In 9.6 seconds he runs 100 yards. In 1 second he runs 100÷9.6 yards. (He runs 9.6 times less in 1 second than in 9.6 seconds) = 10.4+ yards.	9.6)100.00(10.4 96 400 384 16

3. A woman gave a clerk a ten dollar bill ($10.00) in exchange for 7 yards of cloth at $1.25 a yard. How much change did she receive?

Solution

Instruction	*Operation*
Step 1 The problem asks "How much change," but we cannot find this until we find the cost of the cloth. 7 yards cost 7 times 1.25 or $1.25×7=$8.75	**Step 1** $1.25 7 $8.75
Step 2 The amount of change will be the difference between $10.00 and $8.75, which is $1.25	**Step 2** $10.00 8.75 $ 1.25

4. The composition of white metal used in the Navy Department is as follows: In each 100 pounds there are 7.6 pounds of tin, 2.3 pounds of copper, 83.3 pounds of zinc, 3.8 pounds of antimony, and 3.0 pounds of lead. Find the number of pounds of each in 635 pounds of white metal.

Solution

Instruction	*Operation*
Step 1	**Step 1**
We are asked to find the number of pounds of each	$\begin{array}{r} .076 \\ \underline{635} \\ 380 \\ 228 \\ \underline{456} \\ 48.260 \end{array}$
First find number of pounds of tin	
In 100 lb. metal there are 7.6 lb. tin	
In 1 lb. metal there is $7.6 \div 100$ lb. tin $= .076$	
In 635 lb. metal there are $.076 \times 635$ lb. tin $= 48.26$	
Step 2	**Step 2**
Similarly there are	
$.023 \times 635$ lb. copper	$.023 \times 635 = 14.605$ lb.
$.833 \times 635$ lb. zinc	$.833 \times 635 = 528.955$ lb.
$.038 \times 635$ lb. antimony	$.038 \times 635 = 24.13$ lb.
$.030 \times 635$ lb. lead	$.03 \times 635 = 19.05$ lb.

PRACTICE PROBLEMS

1. Find the cost of 24 dozen eggs at $\$.33\frac{1}{3}$ a dozen. Ans. \$8.00

2. A wire fence consists of 16 sections, each 9.75 feet long. How long is the fence? Ans. 156 feet

3. A gallon of water weighs 8.35 pounds. Milk weighs 1.03 times as much as water. What does a gallon of milk weigh? Ans. 8.6+ pounds

4. A line $4\frac{1}{2}$ inches long is divided into 6 equal parts. How long is each part? Ans. .75 inch

5. An automobile traveled 51 miles in an hour (60 minutes). What was the average speed per minute? Ans. 0.85 miles

TRIAL EXAMINATION

Directions. This trial examination is to be used as a preliminary test to see whether you are ready for the final examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems check your answers with the solutions shown on page 28.

If you miss more than two of these problems it means you should review the whole book carefully.

Do not try this trial examination until you have worked every practice problem in this Section.

Do not start the final examination until you have completed this trial examination.

1. (a) Divide 34 by 40.32. (Carry answer to three decimal places.)

(b) Divide 2.3421 by 21.1.

2. (a) Add 37.03+.521+.9+1000+4000.0014.

(b) What is the sum of twenty-six and twenty-six hundredths+seven tenths+six and eighty-three thousandths+four and four thousandths?

3. (a) From 10.0302 subtract two ten-thousandths.

(b) From 900 subtract .009.

4. (a) Reduce $\frac{2}{25}$ to a decimal.

(b) Change .750 to a fraction in its lowest terms.

5. A farm consists of 7 fields, containing $12\frac{3}{4}$ acres, $18\frac{1}{5}$ acres, 9 acres, $24\frac{1}{8}$ acres, $4\frac{7}{8}$ acres, $8\frac{3}{8}$ acres, and $15\frac{3}{5}$ acres. How many acres are there in the farm?

6. What will be the cost of $3\frac{5}{8}$ bolts of cloth, each bolt containing 36.75 yards, at $.85 per yard?

7. An electrical contractor took a job of wiring for $2000. He hired 5 wiremen and 5 helpers to do the job. They worked 8 hours a day. The wiremen cost $1.25 per hour and the helpers 60 cents per hour. In how many days must the men do the job so that the contractor may make a profit of $349.80?

8. Manganese bronze contains the following amounts of metals in each pound: copper .89 pound, tin .10 pound, and manganese .01 pound. How much of each metal is in a propeller which weighs $2378\frac{5}{16}$ pounds. Give results to second decimal.

9. A woman had a bank account of $1754.20. She bought a sewing machine for eighty-nine and three-tenths dollars; a fur coat for three hundred forty-five and four-fifths dollars, and an auto for eight hundred seventy-eight and three-fifths dollars. How much had she left in the bank?

10. In a baseball game one player was at bat 18 times and made 5 hits. What was his average?

FINAL EXAMINATION

1. (a) Add $0.0024+7.023+281.04+.82348$

(b) Add $7.12\frac{3}{4}+.046\frac{1}{2}+75.1\frac{3}{8}+12\frac{5}{16}$

2. (a) Subtract .03456 from .2456

(b) Subtract 14.723 from 157.0032

3. (a) Divide 700 by 6.25

(b) Divide 7.101 by 19 (carry answer to 4 decimal places)

4. Change the following decimals to fractions in lowest terms:

(1) 35.75 (2) 745.125 (3) .3125

5. Change the following to decimal numbers:

(1) $8\frac{3}{8}$ (2) $27\frac{9}{16}$ (3) $15\frac{3}{32}$

6. At the rate of 35.3125 miles per hour, how long will it take to go 347.375 miles? Carry out answer to three decimal places.

7. Three gallons of milk weigh 25.08 pounds. A gallon contains four quarts.

(a) How many quarts in 501.6 pounds?

(b) How much will 501.6 pounds of milk cost at $8\frac{1}{2}$ cents a quart?

8. A grocer has $2\frac{1}{2}$ barrels of A sugar, $5\frac{3}{4}$ barrels of B sugar, $6\frac{1}{4}$ barrels of C sugar, 3.87 barrels of crushed sugar, and 9.89 barrels of pulverized sugar. How many barrels of sugar has he altogether?

9. An iron bar is 18.42 inches long by 2.47 inches wide, by .37 inch thick. (The number of cubic inches is found by multiplying length$\times$width$\times$thickness.) Find the weight of the bar if one cubic inch weighs .261 pound. Give answer correct to three decimal places.

10. A man had two thousand one and a half dollars. He bought an auto for eight hundred fifty dollars and furniture for nine hundred twenty-seven and seven-tenths dollars. How much had he left?

NOTE: To avoid the possibility of having one or more problems wrong review your text and check each problem twice.

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. (a) To divide 34 by 40.32 we set the problem down as in long division.

```
40.32)34    (____
```

The first thing to do is to insert a decimal point after the 4, and to annex two ciphers to the dividend because the second part of the decimal division rule says that we must always annex as many ciphers as are necessary to give the dividend as many decimal places as the divisor. Then we have,

Notice that a decimal point is put in the dividend before the ciphers are annexed.

The rule for dividing decimals says to carry on the actual division just as though the dividend and divisor were *whole numbers and contained no decimal*

We can easily see that 4032 is larger than 3400 so no division is possible until the dividend is made larger. The rule says we can annex as many ciphers as we like. So we annex another cipher to the dividend, making it 34000. Now 4032 can be divided into 34000.

```
40.32)34.00000 (.843 (quotient)
      32 256
       1 7440
       1 6128
         13120
         12096
          1024
```

By trial we find that 4032 goes into 34000 eight times. The remainder is 1744. The 4032 will not divide into the remainder because the remainder is smaller. So we annex another cipher to the dividend, which allows us to annex a cipher to the remainder, making it 17440. The 4032 goes into 17440 four times and leaves a remainder of 1312. We annex another cipher to the dividend and can thus annex a cipher to 1312, making it 13120. The 4032 goes into 13120 about three times.

Now we have 843 as the quotient. We must have three decimal places in the answer so we will test to see if we have enough. The dividend now has five decimal places. The divisor has two. Then $5-2=3$ which is the number of decimal places the quotient will have. When we point off decimal places in the answer we start at the right-hand end and count toward the left. Thus the answer is .843.

Note: If we had wanted five decimal places in the answer we would have annexed enough more ciphers in the dividend so we could have obtained two more numbers in the answer.

1. (b)

```
21.1)2.3421 (.111
     2 11
       232
       211
        211
        211
```

Here it is not necessary to annex any ciphers, because the dividend has enough numbers to start with.

2. (a) To add a group of whole numbers, pure decimals, and mixed decimals, write the numbers as for adding, being careful to keep all the decimal points one directly under the other. For a whole number the decimal point is assumed as being at the right-hand end of the number.

$$\begin{array}{r@{.}l}
37 & 03 \\
 & 521 \\
 & 9 \\
1000 & \\
4000 & 0014 \\
\hline
5038 & 4524 \text{ (Answer)}
\end{array}$$

The adding is done as ordinarily in adding and the decimal point is kept in the same line.

2. (b) To solve this problem it is necessary to remember how to read and write decimals and then add as in part (a) of this problem. Expressing the parts of this problem as pure decimals, mixed decimals, etc., we have

$$\begin{array}{r@{.}l}
26 & 26 \\
 & 7 \\
6 & 083 \\
4 & 004 \\
\hline
37 & 047 \text{ (Answer)}
\end{array}$$

3. (a)

$$\begin{array}{r@{.}l}
10 & 0302 \\
 & 0002 \\
\hline
10 & 0300 \text{ (Answer)}
\end{array}$$

Here the subtrahend is a very small number and changes the minuend but little. If you missed this problem be sure to review the Table of Names and Places in decimals.

3. (b)

$$\begin{array}{r@{.}l}
899 & 99 \\
\cancel{9}\cancel{0}\cancel{0} & \cancel{0}\cancel{0}\cancel{0}^{10} \\
 & 009 \\
\hline
899 & 991 \text{ (Answer)}
\end{array}$$

Here we annexed three ciphers to the minuend to make the subtraction easier to follow. Then we borrowed 1 from the 9 and kept borrowing until the last cipher was made 10. Then we subtracted in the usual manner.

4. (a) To change a fraction to a decimal divide the numerator by the denominator.

$$\tfrac{2}{25} = 25)\overline{2.00}(\underline{.08}$$
$$\underline{2.00}$$

Here the dividend was much smaller than the divisor. Two ciphers had to be annexed before we could divide by 25. In pointing off the answer there were two decimal places in the dividend and none in the divisor, so there must be two in the answer.

4. (b) Following the directions in Lesson 2 we use the figures in the quantity (.750) as the numerator. Omit the decimal point. Then for the denominator use the figure 1 followed by as many ciphers as there are figures in the quantity to the right of the decimal point. Thus we have

$$\frac{750}{1000}$$

Reducing to lowest terms we have

$$\frac{\overset{\overset{3}{\cancel{3}\cancel{0}}}{7\cancel{5}\cancel{0}}}{\underset{\underset{4}{\cancel{4}\cancel{0}}}{\cancel{1}\cancel{0}\cancel{0}\cancel{0}}} - \frac{}{4}$$

5. In order to add up all the fields as mixed decimals it is necessary to change the fractions to decimals. To change fractions to decimals divide the numerator by the denominator. Or, tables such as given in the text can be used.

$$\begin{array}{rcrl}
12\frac{3}{4} \text{ acres} & = & 12.75 & \text{acres} \\
18\frac{1}{5} \text{ acres} & = & 18.2 & \text{acres} \\
9\phantom{\frac{1}{1}} \text{ acres} & = & 9 & \text{acres} \\
24\frac{1}{8} \text{ acres} & = & 24.125 & \text{acres} \\
4\frac{7}{8} \text{ acres} & = & 4.875 & \text{acres} \\
8\frac{3}{8} \text{ acres} & = & 8.375 & \text{acres} \\
15\frac{3}{5} \text{ acres} & = & 15.6 & \text{acres} \\
\hline
 & & 92.925 & \text{total acres.} \quad \text{Answer.}
\end{array}$$

6. First it is necessary to multiply the 36.75 by $3\frac{5}{8}$ in order to find the total number of yards. To do this change $3\frac{5}{8}$ to 3.625. We can do this because $\frac{5}{8} = .625$.

$$\begin{array}{r}
36.75 \\
3.625 \\
\hline
18375 \\
7350 \\
22050 \\
11025 \\
\hline
133.21875
\end{array}$$

To simplify the 133.21875 we can shorten it to 133.22. This can be done because the 8, the first number of the group of numbers we are discarding, is over 5. Then we add 1 to the 21, making it 22. If the first number of the group of numbers being discarded was less than 5 then we would have omitted them

Next multiply 133.22 by \$.85.

$$\begin{array}{r} 133.22 \\ \underline{.85} \\ 66610 \\ 106576 \\ \$113.2370 \end{array}$$

Here we can discard the 70 and add 1 to the 23 for the same reason as explained above. The answer is then \$113.24.

7. First find how much the contractor had to pay each hour for labor.
Each wireman received \$1.25 per hour.
Each helper received \$0.60 per hour.
There were five of each.

$$\begin{array}{r} \text{Therefore } 5\times\$1.25=\$6.25 \\ \text{and } 5\times\$0.60=\underline{3.00} \\ \text{Total cost per hour}=\$9.25 \end{array}$$

The contractor received \$2000.00 for doing the job. He wished to make a profit of \$349.80. Then \$2000.00−\$349.80=\$1650.20. This is the amount he could spend and still make the desired amount of profit.

Now we will divide \$1650.20 by the hourly cost of labor to find how many hours could be used for the job.

$$\begin{array}{l} \underline{9.25})1650.200(\underline{178.4} \\ \underline{925} \\ 7252 \\ \underline{6475} \\ 7770 \\ \underline{7400} \\ 3700 \\ \underline{3700} \end{array}$$

The 178.4 is the number of hours the job is to take.

The men work eight hours a day so—

$$\begin{array}{l} \underline{8})178.4(\underline{22.3}=\text{answer in days.} \\ \underline{16} \\ 18 \\ \underline{16} \\ 24 \\ \underline{24} \end{array}$$

Thus the job must be completed in 22.3 days.

8. $2378\frac{5}{16}=2378.3125$. This is so because $\frac{5}{16}=5\div16=.3125$.

To solve this problem all we have to do is multiply the 2378.3125 by the various amounts (per pound) of the other items.

$$
\begin{aligned}
2378.3125 \times .89 &= 2116.70 \text{ pounds of copper}\\
2378.3125 \times .10 &= 237.83 \text{ pounds of tin}\\
2378.3125 \times .01 &= \underline{23.78} \text{ pounds of manganese}\\
& 2378.31
\end{aligned}
$$

In the multiplication for the copper we shortened the amount from 2116.698125 to 2116.70 by the method explained in the answer to Problem 6. Adding the results gives 2378.31 which is practically the same as 2378.3125 and is, therefore, a check on our work.

9. Change fractions to decimals.

$\$89\frac{3}{10} = \$\ 89.30$

$\$345\frac{4}{5} = \$345.80$

$\$878\frac{3}{5} = \underline{\$878.60}$

$\$1313.70$ she spent

$1754.20 in bank
1313.70
$ 440.50 balance, Answer.

10. This problem is a very common one in sports and is easily solved using the principles of decimals. If the player made 5 hits out of 18 attempts his average was $\frac{5}{18}$ or $5 \div 18 = .278$.

PRACTICAL MATHEMATICS

Section 6

Lesson 1

For Step 1 of this lesson, notice carefully the introductory paragraphs in the lesson. For Step 2, learn the meaning of percentage and the different ways of expressing it. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

PERCENTAGE

In Section 5 you learned that such numbers as .01, .06, .20, and .25 are pure decimals and that they are read one **hundredth,** six **hundredths,** twenty **hundredths,** and twenty-five **hundredths.** In other words, all pure decimals which have two places to the right of the decimal point are called **hundredths.**

One **hundredth** (.01) means 1 part of 100 parts, six **hundredths** (.06) means 6 parts of 100 parts, twenty **hundredths** (.20) means 20 parts of 100 parts, and twenty-five **hundredths** (.25) means 25 parts of 100 parts.

Fig. 1 will make this easier to understand. The whole large square has been divided into 100 small squares. One of the small squares is one **hundredth** of the whole large square. Or, we can say that each of the small squares is a **hundredth** of the large square. The shaded portion $IJLK$ is eight **hundredths** of the large square, since it contains 8 small squares. The shaded portion $ABDC$ is twenty **hundredths** of the large square since it contains 20 small squares. The shaded portion $EFHG$ is twenty-five **hundredths** of the large square since it contains 25 small squares.

In Fig. 1, the shaded areas are **fractional parts** of the whole large square. (You learned about fractional parts in Section 3). In Percentage we think in terms of fractional parts but they are in **hundredths.** From our study of Fig. 1 we have learned how to think of fractional parts in terms of hundredths. Now we can understand the following definitions.

Percentage. Percentage is a name given to a group of rules and methods (used in science and everyday business transactions) in which fractional parts are thought of and used in terms of **hundredths.** In other words Percentage is the process of computing in terms of **hundredths.**

Per Cent. The expression *per cent* is used to indicate the number of **hundredths** being thought of. For example, if we think of the shaded portion *EFHG,* in Fig. 1, we would say 25 **per cent** because this portion contains 25 **hundredths** of the whole square. Thus 25 per cent means the same as .25. If we referred to the shaded portion *ABDC* of the large square we would say 20 per cent, be-

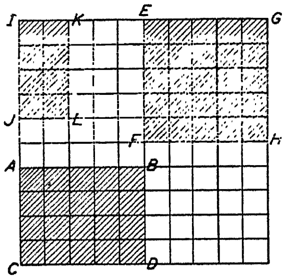


Fig. 1

cause *ABDC* contains 20 small squares or 20 hundredths (.20) of the large square.

The term *per cent* is commonly expressed by the symbol %.

Equivalents. In Section 5 you learned what equivalents are. For example, you learned that the decimal equivalent of $\frac{1}{2}$ is .5 or .50, that the decimal equivalent of $\frac{1}{4}$ is .25, etc. When a fraction and a decimal, or other number, have the same *value* they are said to be equivalent to each other or, simply, equivalents. Equivalents are used frequently in Percentage, so you should become thoroughly acquainted with them and the method by which they are determined.

In the Table of Equivalents many common equivalents are shown in four forms, namely:

(1) Proper fractions
(2) Decimal hundredths
(3) Per cent
(4) Fractional hundredths

You are already familiar with fractions. You learned in Section 5 that a decimal having two places to the right of the decimal

Table of Equivalents

Proper Fractions	Decimal Hundredths	Per Cent	Fractional Hundredths
$\frac{1}{20}$	.05	5%	$\frac{5}{100}$
$\frac{1}{10}$	.10	10%	$\frac{10}{100}$
$\frac{1}{8}$	.12$\frac{1}{2}$	12$\frac{1}{2}$%	$\frac{12\frac{1}{2}}{100}$
$\frac{5}{32}$	.15$\frac{5}{8}$	15$\frac{5}{8}$%	$\frac{15\frac{5}{8}}{100}$
$\frac{1}{6}$	.16$\frac{2}{3}$	16$\frac{2}{3}$%	$\frac{16\frac{2}{3}}{100}$
$\frac{3}{16}$	.18$\frac{3}{4}$	18$\frac{3}{4}$%	$\frac{18\frac{3}{4}}{100}$
$\frac{1}{5}$	.20	20%	$\frac{20}{100}$
$\frac{7}{32}$	.21$\frac{7}{8}$	21$\frac{7}{8}$%	$\frac{21\frac{7}{8}}{100}$
$\frac{1}{4}$	.25	25%	$\frac{25}{100}$
$\frac{9}{32}$	.28$\frac{1}{8}$		$\frac{28\frac{1}{8}}{100}$
$\frac{5}{16}$	.31$\frac{1}{4}$	31$\frac{1}{4}$%	$\frac{31\frac{1}{4}}{100}$
$\frac{1}{3}$	.33$\frac{1}{3}$	33$\frac{1}{3}$%	$\frac{33\frac{1}{3}}{100}$
	.37$\frac{1}{2}$	37$\frac{1}{2}$%	$\frac{37\frac{1}{2}}{100}$
$\frac{2}{5}$	.40	40%	$\frac{40}{100}$
$\frac{7}{16}$	.43$\frac{3}{4}$	43$\frac{3}{4}$%	$\frac{43\frac{3}{4}}{100}$
	.50	50%	$\frac{50}{100}$
	.60	60%	$\frac{60}{100}$
	.62$\frac{1}{2}$	62$\frac{1}{2}$%	$\frac{62\frac{1}{2}}{100}$
	.66$\frac{2}{3}$	66$\frac{2}{3}$%	$\frac{66\frac{2}{3}}{100}$
$\frac{7}{10}$	.70	70%	$\frac{70}{100}$
$\frac{3}{4}$	.75	75%	$\frac{75}{100}$
$\frac{4}{5}$	.80	80%	$\frac{80}{100}$
$\frac{5}{6}$	.83$\frac{1}{3}$	83$\frac{1}{3}$%	$\frac{83\frac{1}{3}}{100}$
$\frac{7}{8}$	.87$\frac{1}{2}$	87$\frac{1}{2}$%	$\frac{87\frac{1}{2}}{100}$
$\frac{9}{10}$	.90	90%	$\frac{90}{100}$

point is called **hundredths.** Such pure decimals are called **decimal hundredths** in percentage. You have just learned that per cent (%) means the number of hundredths being thought of. A fractional hundredth is a common fraction having a denominator of 100 and a numerator equal to the per cent (number of hundredths) being thought of.

Study the $\frac{1}{20}$, .05, 5%, and $\frac{5}{100}$ in the Table of Equivalents. All four of these equivalents have the same value. Therefore if we know one we can easily determine the others. Suppose, for example, we think of 5% (5 per cent) and change it to the other three equivalents.

We know that per cent means the number of **hundredths.** Thus 5% means 5 hundredths. This 5 hundredths means 5 parts of a hundred parts, so we can write it $\frac{5}{100}$. Recalling Section 3 you will remember that the denominator of a fraction indicates into how many parts something has been divided and that the numerator indicates how many of these parts we are thinking of. In percentage we always divide things into **hundredths.** If we say 5% (or 5 hundredths) we are thinking of 5 of the hundredths indicated by the denominator. Thus 5% and $\frac{5}{100}$ are equivalents.

The expression $\frac{5}{100}$ is a fraction, and in a fraction we divide the numerator by the denominator in order to reduce it to a decimal.

$$\begin{array}{r} 100)5.00(.05 \\ \underline{5\ 00} \end{array}$$

The above calculation was done using the rules of Section 5 and proves that $\frac{5}{100}=.05$. Thus 5%, $\frac{5}{100}$, and .05 are equivalents. The $\frac{5}{100}$ is a proper fraction which can be reduced to lowest terms.

$$\frac{\overset{1}{\cancel{5}}}{\underset{20}{\cancel{100}}}=\frac{1}{20}$$ (Review Section 3 if this is not clear.)

From this calculation we see that $\frac{1}{20}$ is also an equivalent of 5%, along with .05 and $\frac{5}{100}$.

In like manner all of the equivalents shown in the Table of Equivalents can be calculated. There are, however, a few items in connection with such amounts as 10, 20, 30, 40, 50, 60, 70, 80, and 90 per cent which you should understand clearly.

Take 20% for example. In changing this to decimal hundredths we divide 20 by 100. Unless you are careful and remember all the rules of Section 5, you might carry on the division like this

$$\begin{array}{r} 100)20.0(.2 \\ \underline{20\ 0} \end{array}$$

Here the answer is only .2 whereas we must have it in terms of hundredths. To keep the correct procedure fresh in your mind remember the rule

Rule 1. *To divide by 100, move the decimal point to the left two places.*

There is no decimal point in the 20 so we simply put the point, following the above rule, in front of the 2, making the answer .20.

Some of the equivalents in the table are easy to change from decimal hundredths, per cent, or fractional hundredths, to proper fractions. For example, as already explained, any **per cent** which ends in 0 or 5, such as 10, 20, 40, 60, 75, etc., can be changed quickly to a proper fraction simply by reducing the fractional hundredths to lowest terms. Per cent expressed as mixed numbers, such as $16\frac{2}{3}$, $83\frac{1}{3}$, $15\frac{5}{8}$, etc., require a little more calculating to reduce to proper fractions. As an example of this take the $15\frac{5}{8}\%$. In terms of fractional hundredths this becomes $\frac{15\frac{5}{8}}{100}$. To reduce this to a proper fraction, change the $15\frac{5}{8}$ to an improper fraction and divide it by 100.

$15\frac{5}{8}=\frac{125}{8}$ (Changing mixed numbers to improper fractions is explained in Section 3)

$$\frac{125}{8}\div 100=\frac{\overset{5}{\overset{\cancel{25}}{\cancel{125}}}}{\underset{32}{\underset{\cancel{160}}{\cancel{800}}}}=\frac{5}{32}$$ (as explained in Section 4)

As another example take $43\frac{3}{4}\%$. Expressed in terms of fractional hundredths it is $\frac{43\frac{3}{4}}{100}$.

$$43\frac{3}{4}=\frac{175}{4}$$

$$\frac{175}{4}\div 100=\frac{\overset{7}{\overset{\cancel{35}}{\cancel{175}}}}{\underset{16}{\underset{\cancel{80}}{\cancel{400}}}}=\frac{7}{16}$$

If you encounter a per cent which is a mixed number not shown in the table, the process is the same. Suppose you had $38\frac{3}{4}\%$ which you wanted to change to a proper fraction equivalent. Expressing

this in terms of fractional hundredths, it is $\frac{38\frac{3}{4}}{100}$. This means to divide $38\frac{3}{4}$ by 100.

$$38\tfrac{3}{4} = \tfrac{155}{4}$$

$$\tfrac{155}{4} \div 100 = \frac{\overset{31}{\cancel{155}}}{80} = \tfrac{31}{80}$$

Per Cent—Smaller or Larger Than 100. When you studied fractions (Section 3) you learned that a whole quantity can be divided into any number of equal parts. You will recall that the number 1 represents one **whole** unit of anything and that all fractions are "parts" of 1. Thus, in fractions, 1 whole unit could be $\frac{2}{2}$, $\frac{10}{10}$, $\frac{25}{25}$, $\frac{100}{100}$, $\frac{125}{125}$, $\frac{200}{200}$, and so on. Each of these fractions equals 1 whole unit because the numerators indicate we are thinking of all the parts into which the object or unit has been divided.

In Percentage we think of a *whole* as being 100 equal parts or 100% or $\frac{100}{100}$. Any per cent larger than 100% is more than 1 whole unit.

Any per cent, such as 7%, 10%, 25%, 55%, etc., which is less than 100%, is less than the whole, and when written in decimal hundredths would have a decimal point in front of it, such as .07, .10, .25, .55, etc.

Any per cent above 100% is more than 1 whole unit. Thus 110%, 125%, 200%, 250%, etc., all mean more than 1 whole. Take the 110% for example. This really means 100% plus 10%, or $\frac{100}{100}+\frac{10}{100}$. Here we have 1 whole unit and a fractional part of another. To write 110% as a decimal we would put a decimal point after the first 1 making it 1.10. In other words we have 1 whole unit plus .10 or 10% of another. Also $110\% = 1\frac{1}{10}$. In like manner

$$125\% = \tfrac{100}{100} + \tfrac{25}{100} = 1.25 = 1\tfrac{1}{4}$$
$$200\% = \tfrac{100}{100} + \tfrac{100}{100} = 2$$
$$250\% = \tfrac{100}{100} + \tfrac{100}{100} + \tfrac{50}{100} = 2.50 = 2\tfrac{1}{2}$$

Per Cent of Quantities. In everyday life we often read or hear such expressions as

(a) Twenty-five per cent of the 200 students had to wear glasses.
(b) Out of every 1,000 automobiles, 40% are over two years old.
(c) Fifty per cent of the American population lives in cities.
(d) Eighty per cent of the world's corn is American grown.
(e) One per cent of your salary goes for Social Security Tax.

From the above typical expressions we can see that in every case it is indicated that some per cent *of* a quantity or amount is to be found before, for example, we know how many students out of every 200 wear glasses.

In (*a*), of the above typical expressions, it is clear that 25% of 200 is to be found. The "of" means the same as "times" or "×." In finding percentages we always change per cent to decimal hundredths. Thus 25% of 200 means $.25 \times 200$. In writing such a problem in the form for multiplication, we always put the quantity (200) over the per cent, thus

$$\begin{array}{r} 200 \\ \underline{.25} \end{array}$$

Rule 2. *To get a per cent of any quantity, change the per cent to decimal hundredths and multiply the quantity by the decimal hundredths. Then point off in the product as many decimal places as there are in the quantity and in the decimal hundredths together.*

ILLUSTRATIVE EXAMPLES

1. What is 8% of $245.50?

Solution. The quantity here is $245.50. You can always tell what the quantity is in Per Cent of Quantity problems because it always follows the "of." Following **Rule 2** we first change 8% to decimal hundredths. This particular per cent is not in the Table of Equivalents so we must figure out what the equivalent is in decimal hundredths. We know the symbol % means number of hundredths. Thus $8\% = 8$ hundredths $= \frac{8}{100}$. Then dividing 8 by 100 we get a decimal hundredths of .08. (Remember **Rule 1.**) Thus $8\% = .08$. Next we multiply the quantity by the decimal hundredths.

$$\begin{array}{rl} \$245.50 & \text{(quantity)} \\ \underline{.08} & \text{(decimal hundredths)} \\ \$19.6400 & \text{(Answer)} \end{array}$$

The multiplying is done as explained in Sections 1 and 5. In pointing off we count the decimal places (starting at the right) in both the quantity and the decimal hundredth. There are two decimal places in each, so there are four in all. Then we point off four places in the answer, starting the counting at the right.

The answer is then \$19.6400. However, as you learned in Section 5, the last two ciphers have no value so they are discarded and the answer is \$19.64.

2. What is 7% of \$842.00.

Solution. This is done exactly as explained for Example 1.

$$\begin{array}{r} \$842.00 \\ .07 \\ \hline \$58.9400 \end{array}$$

The answer is \$58.94.

3. Find 2.7% of 54.

Solution. Rule 2 directed that the per cent be changed to decimal hundredths. This per cent is not in the Table of Equivalents so we must figure it out. We know that per cent means the number of hundredths. Thus 2.7% equals 2.7 hundredths $= \frac{2.7}{100}$. Then to change $\frac{2.7}{100}$ to decimal hundredths we divide 2.7 by 100. (Use **Rule 1.**) When we move the decimal point as many places to the left as there are zeros in the divisor (100) we have

$$2.7 = .027$$

Next we multiply the quantity by .027 and complete the example as explained in **Rule 2** and Example 1.

$$\begin{array}{r} 54 \\ .027 \\ \hline 378 \\ 108 \\ 1.458 \end{array} \text{ (Ans.)}$$

There are no decimal places in the quantity so we count those in the decimal hundredths and point off in the answer as previously explained.

4. Find 1.8% of 195.8.

Solution. By the explanation given in Example 3 we know that 1.8% equals .018 when expressed in decimal hundredths.

$$\begin{array}{r} 195.8 \\ .018 \\ \hline 15664 \\ 1958 \\ \hline 3.5244 \end{array}$$

The answer is 3.5244.

5. A farmer rented a farm on the basis that he would pay 40% of his oats crop as rent. If his total harvest amounted to 8,000 bushels, how many bushels did he pay as rent?

Solution. Here we see that the farmer must pay 40% of all his oats as rent. He harvested 8,000 bushels. Therefore the problem is to find 40% of 8,000. Using **Rule 2.**

$$40\% = .40$$
$$\text{Quantity} = 8{,}000$$

$$\text{Then}\quad \begin{array}{r} 8000 \\ .40 \\ \hline 0000 \\ 32000 \\ \hline 3200.00 \end{array}$$

The farmer paid 3,200 bushels of oats for his rent.

6. A salesman had an agreement with his employer whereby he was paid 12 per cent of his total sales as a commission. If his total sales amounted to \$965 how much commission did he make?

Solution. Using **Rule 1,** the amount is \$965. The per cent, in terms of a decimal hundredths is .12 because 12 per cent $= 12\% = \frac{12}{100} = .12$. Then,

$$\begin{array}{r} \$965 \\ .12 \\ \hline 1930 \\ 965 \\ \$115.80 \end{array}$$

The salesman's commission is \$115.80.

7. Find 135% of 750.

Solution. Previously it was explained that anything over 100, in percentage, is more than a whole. Thus the 135% is 35% more than a whole. Or, we have $\frac{100}{100}+\frac{35}{100}=\frac{135}{100}$. If we divide 135 by

100 we have 1.35. Therefore 135% expressed as a decimal is 1.35. Then

$$\begin{array}{r} 750 \\ 1.35 \\ \hline 3750 \\ 2250 \\ 750 \\ \hline 1012.50 \end{array} \text{ (Answer)}$$

Here the answer is larger than the original quantity. We can reason this out easily because 100% means a *whole* or $\frac{100}{100}$ and $\frac{100}{100}=1$ because the denominator divides exactly once into the numerator. Then $1\times750=750$. Now if we take 35% of 750 we have $750\times.35=262.50$. If we add the $750+262.50$ we get 1012.50, which proves our first answer.

8. Find 220% of 348.

Solution. $220\%=2.20$ in decimal hundredths. This is obtained by dividing 220 by 100.

$$\begin{array}{r} 348 \\ 2.20 \\ \hline 000 \\ 696 \\ 696 \\ \hline 765.60 \end{array} \text{ (Answer)}$$

9. Find $21\frac{7}{8}\%$ of 240.

Solution. Where the per cent is a mixed number, the process of finding the percentage is different. We know that $21\frac{7}{8}\%=.21\frac{7}{8}$. Thus far we have followed **Rule 2** as explained in the previous examples. The difference comes in the process of multiplying.

$$\begin{array}{ll} 1. & 240 \\ 2. & .21\frac{7}{8} \\ \hline 3. & 210\leftarrow \\ 4. & 240 \\ 5. & 480 \\ \hline 6. & 52.50 \end{array}$$

The 210 is the result of multiplying 240 by $\frac{7}{8}$. Thus

$$\frac{7}{\cancel{8}}\times\overset{30}{\cancel{240}}=\frac{210}{1}=210$$

In the above calculation the first step is to multiply the quantity

(240) by $\frac{7}{8}$. This is done as explained in Section 4. The product (210) is put down as shown in Line 3. From here on the multiplication is the same as usual except that the result of multiplying the 240 by 1 (Line 4) is not moved one place to the left. The result of the next multiplication (Line 5) is moved over one place as is usual in multiplication. Pointing off is done as previously shown.

10. Find $83\frac{1}{3}\%$ of 235.

Solution. Multiplication is done as explained for Example 9.

$83\frac{1}{3}\% = .83\frac{1}{3}$

$$
\begin{array}{r}
235 \\
\underline{.83\frac{1}{3}} \\
78 \leftarrow \\
705 \\
\underline{1880} \\
195.83
\end{array}
$$

This 78 is the result of multiplying 235 by $\frac{1}{3}$. Thus

$$\tfrac{1}{3} \times 235 = \tfrac{235}{3} = 78\tfrac{1}{3}.$$

The $\frac{1}{3}$ is less than half of 1 so it can be dropped.

11. Find $83\frac{1}{3}\%$ of 236.

Solution

$83\frac{1}{3}\% = .83\frac{1}{3}$

$$
\begin{array}{r}
236 \\
\underline{.83\frac{1}{3}} \\
79 \leftarrow \\
708 \\
\underline{1888} \\
196.67
\end{array}
$$

The 79 is the result of multiplying 236 by $\frac{1}{3}$. Thus

$$\tfrac{1}{3} \times 236 = \tfrac{236}{3} = 78\tfrac{2}{3}.$$

This $\frac{2}{3}$ is over half of 1 so it is dropped and 1 added to 78 making it 79.

Note: After you have studied the eleven examples given, see if you can work them without looking at the solutions.

PRACTICE PROBLEMS

After you have worked the following problems, turn to page 46 and compare your answers with the correct answers.

1. Find 7% of 100.
2. Find 13% of 185.
3. Find 22% of 16.2.
4. Find 5.3% of 75.
5. Find $37\frac{1}{2}\%$ of 90.
6. Find $62\frac{1}{2}\%$ of 624.
7. Find $66\frac{2}{3}\%$ of 495.
8. Find $15\frac{5}{8}\%$ of 1632.
9. Find 18% of 750.
10. Find 32% of 1452.
11. Find 125% of 355.
12. Find 8.7% of 135.
13. Find 200% of 700.
14. Find 47% of 100.

15. A carpenter estimated the cost of a building to be $8,000. If he added 15% for profit and overhead, what was the total cost?

16. If a residence cost $10,000 and if 26% of that cost was for masonry, how much did the masonry work cost?

Summary. Sometimes the product, in decimal or percentage calculations, has more than two places to the right of the decimal point. For example, if we find 16% of 223.23 the answer is 35.7168. Here there are four places to the right of the decimal point. Unless strict accuracy is required, the four places can be reduced to two or three, as explained in Section 5.

Suppose only three places to the right of the decimal are required. The 8 is the fourth place. It is more than 5 so when we remove the 8 we add 1 to the 6. Then the number becomes 35.717−.

Suppose the accuracy requirements called for only two places. The 68 would be removed. It is over 50, so we would add 1 to the 1. The number would become 35.72−.

The minus (−) signs indicate that the number to the right of the decimal is really not quite what is shown because, taking 35.72− for example, the .72 is really .7168 and .72 is larger than .7168. We called it equal to .72 for convenience. Thus the minus sign brings this to attention. It is not always necessary to show a minus sign.

Now suppose we had a percentage figure like 35.7421. If we wanted to show only two places to the right of the decimal point, we would take off or remove the 21. This 21 is less than 50, so nothing can be added to the 4. However, we can write the shortened number 35.74+. The plus (+) sign indicates that the exact answer has more places, or is a little larger, than 35.74.

Note: So as to avoid any confusion, the foregoing examples and problems, and those to come, are shown with their full answers.

Short Way of Finding Per Cent of Quantities. In the previous explanations you learned how to find a percentage of a quantity by multiplication. Thus 25% of 500 is

$$\begin{array}{r} 500 \\ .25 \\ \hline 2500 \\ 1000 \\ \hline 125.00 \end{array} \text{ (Answer)}$$

From what you learned about equivalents, you can quickly find that 25% has $\frac{1}{4}$ as one of its equivalents. This is shown in the Table of Equivalents too. Then to find 25% of 500 we can simply find $\frac{1}{4}$ of 500.

$$\begin{array}{r} 4\overline{)500} \\ \hline 125 \end{array} \text{ (Answer)}$$

If the equivalent fraction has a numerator above 1 in value, such as $\frac{2}{5}$, the process is as follows. Suppose we want to find 40% of 360. By calculation or by the Table of Equivalents, we know that $\frac{2}{5}$ is the equivalent fraction of 40%.

$$\begin{array}{r} 5\overline{)360} \\ \hline 72 \end{array}$$

Now, because the fraction had a 2 for its numerator, we must multiply the 72 by 2. Thus $2 \times 72 = 144$, Answer.

ILLUSTRATIVE EXAMPLES

1. Find 20% of 135.

Solution. From the equivalent table $20\% = \frac{1}{5}$.

$$\begin{array}{r} 5\overline{)135} \\ \hline 27 \end{array}$$

In the above we simply divided 135 by 5 using short division. The answer is 27.

2. Find $62\frac{1}{2}\%$ of 576.

Solution

$$62\tfrac{1}{2}\% = \tfrac{5}{8}$$

$$\begin{array}{r} 8\overline{)576} \\ \hline 72 \\ 5 \\ \hline 360 \end{array} \text{ (Answer)}$$

Here we divided 576 by 8 and multiplied the quotient by the numerator 5.

3. Find $12\frac{1}{2}\%$ of 8464.

Solution

$$12\tfrac{1}{2}\% = \tfrac{1}{8}$$

$$\begin{array}{r} 8\overline{)8464} \\ \hline 1058 \end{array} \text{ (Answer)}$$

PRACTICE PROBLEMS

After you have worked all of the following problems, compare your answers with the correct answers shown on page 46.

Find the following percentages using the short method.

1. 5% of \$40.00.
2. 10% of 600 bushels.
3. $16\frac{2}{3}$% of 234 feet.
4. 25% of 724 acres.
5. $33\frac{1}{3}$% of 903 rods.
6. 40% of 155 gallons.
7. $37\frac{1}{2}$% of 240 miles.
8. 50% of \$723.
9. $33\frac{3}{4}$% of 640.
10. $74\frac{4}{5}$% of 1250.
11. $87\frac{1}{2}$% of 256.
12. $15\frac{5}{8}$% of 480.
13. 20% of 500.
14. 80% of 60.
15. 70% of 850.

REVIEW OF LESSON 1

Before taking up this review you should have studied Lesson 1 and have worked all of the Illustrative Examples and Practice Problems.

This review is to give you a means of testing your knowledge of Lesson 1. Read the questions carefully and answer them *without looking at Lesson 1.* Answer the questions on paper just as if you were writing out an examination.

After you have written out answers to all of the questions in this review, turn to page 46 and compare your solutions with those shown. If any of your answers are wrong, review Lesson 1 thoroughly.

You correct your own work. Your instructor does not have to see your work.

1. What is the difference between the terms **hundred** and **hundredths?**

2. What is another name for the expression, "the number of hundredths."

3. How would you find 250% of 350?

4. Give the proper fraction, decimal hundredths, and fractional hundredths equivalents for

(a) $37\frac{1}{2}$%
(b) $13\frac{1}{3}$%

5. Find 8.6% of 256. Show by calculations how you changed 8.6% into decimal hundredths. Prove that your decimal hundredth is correct.

6. Explain how to change the following to decimal hundredths.
   (a) 9%
   (b) 6.7%
   (c) 12%
7. Change 16.7% to decimal hundredths.
8. What is the difference between 150% and 1.5%?
9. Change $\frac{80}{100}$ to its proper fraction equivalent.
10. Show two methods of finding $83\frac{1}{3}$% of 366.

Lesson 2

For Step 1, keep in mind the principles, rules, and methods you learned in Lesson 1. For Step 2, learn the new principles of this lesson. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

PERCENTAGE (continued)

In Lesson 1 you learned what *Percentage* and *Per Cent* mean. You learned also that in percentage we think entirely in terms of *hundredths*. Then you learned how to express per cent in terms of hundredths and, finally, how to find *per cent of quantities*. This was the main part of the lesson, and you learned several ways of doing this. Be sure you understand Lesson 1 before you start this lesson.

Finding What Per Cent One Number Is of Another. In Lesson 1 you learned how to find any per cent of any quantity. Now you will learn how to find what per cent one number is of another number. In other words you will learn to compare numbers by **hundredths.**

You have often read or heard typical examples of finding what per cent one number is of another. Several are given in the following.

(*a*) What per cent of 40 is 8?

(*b*) If a man earns $400 a month and spends $120 for rent, what per cent of his income goes for rent?

(*c*) A boy saved $21 toward buying a $30 bicycle. What per cent of the price has he saved?

(*d*) If you planned a motor trip of 50 miles, what per cent of the trip had you completed after going 4 miles?

From the above typical expressions we can see that a per cent is required. In other words we must find the per cent.

In Section 3 we learned that the denominator of a fraction indicated into how many parts some object had been divided, and that the numerator indicated how many of these parts we were thinking of.

Now take example (*a*) above. Here we want to find what per cent 8 is of 40. In this case we can think of 40 being the denominator because it is the total number of parts. We can think of 8 being the numerator because it is the number of parts we are thinking of.

Thus example (*a*) could be written as,

$$\frac{8 \text{ (Parts we are thinking of) or (numerator)}}{40 \text{ (Total parts) or (denominator)}}$$

As a further illustration take example (*b*). Here the \$400 is the total parts, or denominator, and the \$120 is the number of parts we are thinking of, the numerator. In example (*c*) the \$30 is the total number of parts, or the denominator, and the \$21 is the number of parts we are thinking of, or the numerator.

To find what per cent one number is of another use the following Rule.

Rule 3. *To find what per cent one number is of another, express the numbers as numerator and denominator and divide the numerator by the denominator. Divide and point off in the same way you learned for changing fractions to decimal numbers in Section 5.*

ILLUSTRATIVE EXAMPLES

1. If you plan a motor trip of 50 miles, what per cent of the trip have you completed after traveling 4 miles?

Solution. The 50 miles is the total number of miles, or total number of parts, so it becomes the denominator. The 4 miles is the number of miles we are thinking of, or numerator. Thus we have

$$\frac{4}{50}$$

Following **Rule 3** we divide the numerator by the denominator. This is done following the rules you learned in Section 5 relative to changing fractions to decimal numbers.

$$\frac{4}{50} = 50\overline{)4.00}(.08$$
$$\underline{4\ 00}$$

We know from Lesson 1 that $.08 = 8\%$. Thus the answer is 8%.

Proof

$$8 \text{ per cent of } 50 = .08 \text{ of } 50$$

$$\begin{array}{rr} \text{Then} & 50 \\ & \underline{.08} \\ & 4.00 \end{array}$$

2. One day 8 pupils out of a class of 40 were absent. What per cent were absent?

Solution. The 40 pupils is the total number of pupils, or total number of parts, so it becomes the denominator. The 8 pupils is the number of pupils we are thinking of, or the numerator.

Thus we have

$$\frac{8}{40}$$

Following **Rule 3**

$$\frac{8}{40} = 40)8.0(.2$$
$$80$$

The answer is .2, but in percentage we think in terms of *hundredths* so we change the .2 to .20. We know that $.20 = 20$ per cent. (Answer)

Proof

$$20 \text{ per cent of } 40 = .20 \text{ of } 40$$

$$\begin{array}{rr} \text{Then} & 40 \\ & \underline{.20} \\ & 00 \\ & \underline{80} \\ & 8.00 \end{array}$$

3. A boy saved \$21.00 toward buying a \$30.00 bicycle. What per cent of the price had he saved?

Solution. The \$30.00 is the total number of dollars, or total number of parts, so it becomes the denominator. The \$21.00 is the number of dollars we are thinking of, or numerator.

Thus we have

$$\frac{\$21.00}{\$30.00}$$

Following **Rule 3**

$$\frac{\$21.00}{\$30.00} = 30.00)21.000(.7$$
$$\underline{21\ 000}$$

As in Example 2, the .7 becomes .70 in percentage. Then $.70 = 70$ per cent. (Answer)

Proof

$$70 \text{ per cent of } \$30.00 = .70 \text{ of } \$30.00$$

$$\begin{array}{lr} \text{Then} & \$30.00 \\ & .70 \\ \hline & 0000 \\ & 21000 \\ & \$21.0000 = \$21.00 \end{array}$$

4. What per cent of 50 is 50?

Solution. As explained in previous examples, this example would be written $\frac{50}{50}$.

$$\frac{50}{50} = 50)50(1 = 100 \text{ per cent}$$
$$50$$

The 1 has no decimal place so it is a whole unit. In percentage a whole unit is 100%. Therefore the answer to this example is 100 per cent.

PRACTICE PROBLEMS

After you have worked all of the following problems, compare your answers with the correct answers shown on page 48.

1. What per cent of 130 is 19.50?
2. What per cent of 500 is 200?
3. What per cent of 500 is 250?
4. What per cent of 780 is 195?
5. A boy worked during the summer and made \$240. He saved \$160. What per cent of his earnings did he save?
6. In a box of 840 bolts 70 were defective. What per cent were defective?
7. A school had 720 students. In 6 months, enrollment increased 108. What per cent was the increase over the first number listed?
8. A contractor agreed to build 20 miles of new road. What per cent of his work had he completed after he had built 15 miles?
9. A skyscraper was to be 40 stories high. What per cent of the height was complete after 16 stories were finished?
10. A town of 4,000 had a public school enrollment of 800. What per cent of the town's population were going to public school?

Short Method of Finding What Per Cent One Number Is of Another. Suppose we wanted to find what per cent 8 was of 40. We can apply the first part of Rule 3 and have $\frac{8}{40}$. Instead of carrying on long division we can reduce to lowest terms.

$$1$$
$$\cancel{2}$$
$$\cancel{4}$$
$$\frac{\cancel{8}}{\cancel{40}}=\frac{1}{5}$$ Here we cancelled, using 2.

$$\cancel{10}$$
$$5$$

From the Table of Equivalents we know that $\frac{1}{5}$ is the proper fraction equivalent for 20%. Therefore 8 is 20 per cent of 40.

ILLUSTRATIVE EXAMPLES

1. What per cent of 60 is 15?

Solution. Using the first part of Rule 3 we have $\frac{15}{60}$ Reducing to lowest terms, we have

$$1$$
$$\frac{\cancel{15}}{\cancel{60}} =$$

From the Table of Equivalents we see that $\frac{1}{4}$ is the fraction equivalent of 25%. So 15 is 25 per cent of 60.

2. A man's monthly income is $400. He spends $120 per month for rent. What per cent of his income goes for rent?

Solution. Using the first part of Rule 3 we have $\frac{120}{400}$, and reducing to lowest terms we have

$$3$$
$$\cancel{15}$$
$$\cancel{30}$$
$$\cancel{60}$$
$$\frac{\cancel{120}}{\cancel{400}}=\frac{3}{10}$$
$$\cancel{200}$$
$$\cancel{100}$$
$$\cancel{50}$$
$$10$$

This $\frac{3}{10}$ is not shown in the Table of Equivalents, so we change it to a decimal number by dividing numerator by denominator.

$$\begin{array}{r} 10\overline{)3.0}(.3 \\ \underline{3\,0} \end{array}$$

We know that .3 is .30 in terms of hundredths. Also $.30 = 30\%$. Therefore the answer is 30 per cent.

PRACTICE PROBLEMS

(Use Short Method)

After you have worked the following problems, compare your answers with the correct answers shown on page 48.

1. What per cent of 250 is 100?
2. What per cent of 380 is 342?
3. What per cent of 860 is 516?
4. What per cent of 900 is 270?
5. What per cent of 1020 is 714?
6. What per cent of 520 is 416?

Finding a Number or Quantity When a Per Cent of It Is Given. So far we have learned how to find the per cent of a quantity and what per cent one number is of another. Now we will learn how to find a quantity when the per cent is known.

You have often encountered such typical examples as follow.

(*a*) If you paid the sale price of \$8 for a sweater which had been marked down 20%, what was the regular price of the sweater?

(*b*) 7 is 25% of what number?

(*c*) If you planned to save 9% of your salary for one year and by this plan actually saved \$270, what must your salary be?

From the above examples you can see that in every case it is necessary to find the quantity. This is the reverse of finding a per cent of a quantity.

To better illustrate the process of finding a number when the per cent is given, study the typical example (*b*). Here we want to know what number 7 is 25% of. We know that 25% of the number is 7. Then 1% of the number would be $7 \div 25 = .28$. If 1% of the number is .28 then the number would be $.28 \times 100 = 28$. This reasoning is simple and can be proved just as easily. If we multiply 28 by .25 the answer is 7, which proves the above reasoning.

Rule 4. *To find a number or quantity when a per cent is given, first determine what 1% of the required number is by dividing the given number by the given per cent. Then multiply this quotient by 100 to find the number.*

ILLUSTRATIVE EXAMPLES

1. 8 is 20% of what number?

Solution. Following **Rule 4** we first find what 1% is by dividing 8 by 20.

$$\underline{20})8.0(\underline{.4}$$
$$\underline{8\,0}$$

Next, also following the rule, multiply .4 by 100.

$$.4 \times 100 = 40 \text{ (Answer)}$$

Proof

$$40 \times .20 = 8$$

2. 2 is 50% of what number?

Solution. Following **Rule 4** we first find what 1% is by dividing 2 by 50.

$$\underline{50})2.00(\underline{.04}$$
$$\underline{2\,00}$$

Next, also following the rule, multiply .04 × 100.

$$.04 \times 100 = 4 \text{ (Answer)}$$

Proof

$$4 \times .50 = 2$$

3. If there are 12,000 school children in a certain city and if 20% of the population are school children, what is the population of the city?

Solution. Studying this example, we can see that it means the same as asking, "12,000 is 20% of what number?"

So, using **Rule 4** to find 1% we have

$$\underline{20})12000(\underline{600}$$
$$\underline{120}$$
$$00$$

Then multiply by 100.

$$600 \times 100 = 60{,}000 \text{ (Answer)}$$

Proof $$60{,}000 \times .20 = 12{,}000.$$

PRACTICE PROBLEMS

After you have completed the following problems, refer to page 48 and compare your answers with the correct answers shown.

1. 1 is 10% of what number?
2. 16 is 4% of what number?
3. 20 is 5% of what number?
4. 38 is 2% of what number?
5. 27 is 9% of what number?
6. 120 is 15% of what number?
7. If a man donated $32 to the Community Chest and this amount is 1% of his yearly salary, what is his yearly salary?
8. A group of employees decided to raise a benefit fund. After a campaign of one week $5000 had been collected, which was 50% of the required sum. How much was required?

REVIEW OF LESSONS 1 AND 2

In Lessons 1 and 2 you have learned the following items:

1. The meaning of percentage.
2. The meaning and importance of per cent.
3. The way to express per cent in terms of hundredths.
4. How to picture per cent (Fig. 1).
5. How to find any per cent of any quantity.
6. How to use equivalents to find per cent of quantities by the short method.
7. How to find what per cent one number is of another.
8. Short method for doing Item 7.
9. How to find numbers or quantities when a per cent is given.
10. How to solve everyday percentage problems.

Go over these items carefully to make *absolutely sure* you understand them. Explain each item thoroughly to yourself. If possible explain each item aloud to a friend. Review the lessons if every item is not perfectly clear to you.

The 10 items of the above summary of things you have learned in Lessons 1 and 2 have prepared you to solve three distinct kinds of percentage problems; namely, Items 5, 7, and 9. This review is given to help you make sure you can solve typical problems in these three groups. The following examples and diagrams will help you to fix these three types of problems more securely in your mind.

ILLUSTRATIVE EXAMPLES

1. A man planned to drive his auto to a city 500 miles away. After he had gone 8% of the way, how many miles had he traveled?

Solution. The following diagram shows what is known and what is required in this example.

Whole distance	%	Part covered
500 miles	8%	? miles

Here we know the whole number of miles (quantity) and the per cent. The distance covered is to be found. In other words we must find 8% of 500. Therefore we use **Rule 2.**

$$8\% = \frac{8}{100} = .08$$

$$\begin{array}{r} 500 \\ .08 \\ \hline 40.00 \end{array}$$

Answer equals 40 miles.

2. A man planned to drive his auto to a city 500 miles away. After he had gone 100 miles, what per cent of the trip had he completed?

Solution. The following diagram shows what is known and what is required in this example.

Whole distance	%	Part covered
500 miles	?	100 miles

Here we know the whole distance (quantity) and the part covered. The per cent is to be found. Therefore we use **Rule 3.** The 500 miles is the total number of miles or total number of parts, so it becomes the denominator. The 100 miles is the number of miles we are thinking of, so this becomes the numerator.

Thus $\frac{100}{500}$

Then $500)100.0(.2$

$100\ 0$

We know that .2 expressed in terms of hundredths is .20. Therefore the answer is 20%.

3. After a man had driven his auto a distance of 75 miles, he had traveled 15% of the whole distance he planned to go. How far had he planned to go?

Solution. The following diagram shows what is known and what is required in this problem.

Whole distance	%	Part covered
? miles	15%	75 miles

Here we know the part covered and the per cent. The whole distance (quantity) is to be found. Therefore we use **Rule 4.**

$$\text{Thus}\quad 15)75(5$$
$$75$$

$$\text{Then}\quad 5\times 100=500$$

Answer equals 500 miles.

REVIEW PROBLEMS

After you have solved all of the following problems, compare your answers with the correct answers shown on page 48.

1. If 1,680 out of 3,500 pupils are boys, what per cent of the pupils are boys?

2. Out of 3,500 pupils in a school, 25% were found to be underweight. How many children were underweight?

3. A baseball team won 90% of the 40 games it played. How many games did it win?

4. A man drove his car 32 miles. At this point he had driven 40% of the way. How far was he going?

5. A farmer lost 10 out of every 40 bushels of potatoes by decay. What per cent did he lose?

6. A merchant failing in business pays 45% to his creditors. How much did he owe, if at this rate he paid out $1,125?

7. If you owed $3,000 and paid $2,500, what per cent did you pay?

8. How much was saved by buying a tire at a reduction of 30% if the regular price was $20?

9. An increase of 15% in the price of an automobile amounted to $240. What was the original price?

10. 8% of what number is 240?

11. 32% of what number is 384?

12. 45% of what number is 360?

Lesson 3

For Step 1, keep in mind that under every new topic names are used that have definite meanings and that you must know before any progress can be made in your study. For Step 2, study the various applications of the rules. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

PERCENTAGE IN BUSINESS

In Lessons 1 and 2 you learned the basic principles of percentage and many of the most common uses for it. In this lesson you will learn how percentage is used in the many kinds of everyday business transactions.

For example

A business man sometimes borrows money from a bank. For using this money a definite length of time he must pay **interest** which is a certain per cent of the money borrowed.

Stores and other sales establishments sometimes offer **discounts** at special sales in order to attract more buyers.

Salesmen work on a **commission** basis whereby they receive a certain per cent of all the money they take in.

Merchants who import cigars, rugs, cameras, etc. from other countries pay **duties** at a certain per cent.

Property owners pay **taxes** which are calculated in per cent.

Premiums are figured in percentages for all forms of **insurance.**

As you study the following explanations of these typical uses of percentage in business, you should notice that **Rules 5, 6** and **7** are developed from **Rules 2, 3,** and **4,** which you studied in Lessons 1 and 2. That is, the rules we will use for Interest, Discounts, Commission, etc., calculations are the same as the rules learned previously, except for the addition of one more simple calculation.

INTEREST

If you rent a house you must pay the owner of the house a certain monthly sum of money which is commonly called **rent.** In other words, you pay the owner for the privilege of using his property. In the same way, if you borrow money from a bank you must pay the bank a certain amount for the privilege of using the money. What you pay the bank in addition to the money you borrowed is called **interest.**

Suppose you borrow $500 from a bank. Before you receive the money you must sign a "note." This note shows the amount of money you are borrowing, the date on which the amount borrowed is to be paid back, and the rate of interest. The following definitions explain the various terms used in such a transaction.

Principal. The money being borrowed is called the principal. Thus if you borrow $500 this amount is the principal, and it is the sum of money upon which the interest is paid.

Interest. As previously explained, interest is the money paid for the privilege of using money belonging to another person or business organization. If you borrow $500, as outlined above, the note you sign may show that you must pay interest at the rate of 6% per year. This means that you will pay 6% of $500. This rate of 6% can be thought of as $6 per hundred dollars. We know 6% equals .06. We use **Rule 2** to find the interest for one year on $500. Thus

$$\$500 \times .06 = \$$$

The $30 is **interest** which you pay for the privilege of using the $500 for one year. You will find that interest rates vary and generally run from 1% to 7%.

Time. When you borrow money, the interest is calculated **per year** by using whatever rate of interest is called for (such as 6%) and **Rule 2.** Thus, in the above, we saw that interest for one year on $500 at 6% equals $30. If the loan runs for two years, we multiply $30 by 2. If the loan runs for 3 years, we multiply $30 by 3, etc. If the $500 loan, at 6%, is for only 6 months ($\frac{1}{2}$ year) we divide $30 by 2. Thus **time** means the life of the loan.

Amount. If you borrow $500 at 6% interest for one year, when the time comes to pay the loan you pay $500 plus $30 interest, or a total of $530. The total ($530) that you would pay is called the **amount.**

To Find the Interest. Now we can state a rule to follow in calculating interest.

Rule 5. *To find interest for one year, multiply the principal (amount borrowed) by the rate of interest. This is done exactly as finding a per cent of a quantity (Rule 2). If the interest is for more than one year, multiply the interest for one year by the number of years.*

ILLUSTRATIVE EXAMPLES

1. Find the interest, at 5%, on $850 for one year.

Solution.

The principal is $850 (money loaned).

The rate of interest is 5%. We know $5\% = \frac{5}{100} = .05$. Then, following **Rule 5**

$850	(principal)
.05	(rate of interest)
$42.50	(interest for one year)

Pointing off in the answer is done as explained in **Rule 2.**

2. Find the interest, at 6%, on $1,250 for five years.

Solution

The principal is $1,250.

The rate of interest is 6%. Then

$1250	(principal)
.06	(rate of interest)
$75.00	(interest for one year)
5	(time)
$375.00	(interest for 5 years)

3. Find the interest, at $5\frac{1}{2}\%$, on $2,500 for 6 years plus 6 months. What is the amount?

Solution

$2500	(principal)
.055	(rate of interest)
12500	
12500	
$137.500	(interest for one year)

We know $5\frac{1}{2}\% = 5.5\%$ (See Section 5). We know that $5.5\% = \frac{5.5}{100}$ and that $\frac{5.5}{100} = .055$. (See Rule 1)

We know that $137.500 = $137.50. The last zero in 137.500 does not mean anything in percentage because we think in terms of hundredths.

$137.50	(interest for one year)
6.5	(time in years) . . . .
68750	
82500	
$893.750	(interest for $6\frac{1}{2}$ years)

6 months $= \frac{1}{2}$ year. $6\frac{1}{2}$ years $= 6.5$ years. See Section 5

Here again the zero in the .750 means nothing and can be discarded. Thus the answer is \$893.75. The amount is \$2,500+\$893.75= \$3,393.75.

To Find the Principal. Sometimes we know the interest, rate of interest, and time, and want to find the principal. In **Rule 4** you learned how to find a number or quantity when a per cent of it is given. In interest calculations the *interest* and *rate of interest* correspond to the *given number* and the *given per cent* of **Rule 4.** Then we can use **Rule 6,** which is **Rule 4** expressed in different words.

Rule 6. *To find the principal when interest, rate of interest and time are known: (a) Find the interest for 1 year by dividing total interest by number of years. (b) Divide quotient by the given rate of interest. (c) Multiply this quotient by 100.*

ILLUSTRATIVE EXAMPLES

1. The interest for 3 years @5% is \$52.50. Find the principal.

Solution. The \$52.50 is the total interest on the unknown principal for 3 years. The interest for one year would be

(*a*) $\$52.50 \div 3 = \$17.50$ (interest for 1 year)

We know that $\$17.50 = 5\%$ of the principal. Then

(*b*) $\$17.50 \div 5 = \$3.50$ (1% of principal)

(*c*) $\$3.50 \times 100 = \$350$ (principal)

Proof. If we apply **Rule 5** we have

$\$350 \times .05 = \$17.50$ (interest for 1 year)

$\$17.50 \times 3 = \$52.50$ (interest for 3 years)

2. The interest for 2 years plus 4 months at $7\frac{1}{2}\%$ was \$220.50. Find the principal.

Solution. We know that 2 years and 4 months is the same as $2\frac{1}{3}$ years. Then, using **Rule 8** of Section 4,

(*a*) $\$220.50 \div 2\frac{1}{3} = \$94.50$ (interest for 1 year)

We know that $\$94.50 = 7\frac{1}{2}\%$ of the principal. Then, again using **Rule 8** of Section 4,

(*b*) $\$94.50 \div 7\frac{1}{2} = \$12.60$ (1% of principal)

(*c*) $\$12.60 \times 100 = \$1260$ (principal) Answer

Proof

$\$1260 \times .075 = \$94.50$ (interest for 1 year)

$\$94.50 \times 2\frac{1}{3} = \$220.50$ (interest for $2\frac{1}{3}$ years)

To Find the Rate of Interest. Sometimes the principal, time, and interest are given, and we want to find the rate of interest.

Rule 7. *To find the rate of interest when principal, time and interest are given: Find interest for one year by dividing total interest by number of years; then divide this quotient by the principal.*

ILLUSTRATIVE EXAMPLES

1. The interest on \$1120 for $3\frac{1}{2}$ years is \$156.80. Find rate of interest.

Solution. The \$156.80 is the interest for $3\frac{1}{2}$ years. We know that $3\frac{1}{2}=3.5$. Then the interest for one year would be

```
3.5)156.80(44.8
     140
     ---
      168
      140
       280
       280
       ---
```

(We know that we can add a cipher to the answer without changing its value, to make it \$44.80 = interest for 1 year)

We now must find what per cent \$44.80 is of \$1120 (the principal). Following **Rule 7,** we divide the quotient, \$44.80, by the principal, \$1120. This is done by following **Rule 3**

$$\frac{\$44.80}{\$1120} = \begin{array}{l} 1120)44.80(.04 \\ \quad\;\; 44\,80 \end{array}$$

The answer is .04 or 4% = rate of interest (Answer)

Proof

$$\$1120 \times .04 = \$44.80 \text{ (interest for 1 year)}$$

$$\$44.80 \times 3\tfrac{1}{2} = \$156.80 \text{ (interest for 3 years)}$$

2. The interest on \$5,000 for 6 years is \$1,500. Find rate of interest.

Solution. The \$1,500 is the interest for 6 years. Then the interest for 1 year would be

$$\$1500 \div 6 = \$250 \text{ (interest for 1 year)}$$

We now must find what per cent \$250 is of \$5,000. Following our rule, we divide the quotient, \$250, by the principal, \$5,000.

$$\frac{\$250}{\$5000} = 5000)250.00(.05$$

The answer is .05, or 5% = rate of interest (Answer)

Proof

$$\$5{,}000 \times .05 = \$250 \text{ (interest for 1 year)}$$
$$\$250 \times 6 = \$1{,}500 \text{ (interest for 6 years)}$$

To Find the Time. Sometimes we know the principal, interest, and rate of interest and want to determine the time.

Rule 8. *To find the time when principal, interest and rate are given, multiply the principal by the rate of interest. This gives interest for one year. Then divide total interest by interest for one year.*

ILLUSTRATIVE EXAMPLE

In what time will \$275 gain \$55 interest at 6%?

Solution

$$\begin{array}{r} \$275 \\ \underline{.06} \\ \$16.50 \end{array} \text{ (interest for one year)}$$

$$\text{Then } \underline{16.50})55.0000(\underline{3.33\tfrac{1}{3}} = 3\tfrac{1}{3}$$
$$\begin{array}{r} \underline{4950} \\ 5500 \\ 4950 \\ 5500 \\ \underline{4950} \\ \dfrac{550}{1650} = \dfrac{1}{3} \end{array}$$

We know $.33\frac{1}{3} = \frac{1}{3}$ (see Table of Equivalents).

Time is therefore $3\frac{1}{3}$ years.

Proof

$$\$275 \times .06 = \$16.50$$
$$\$16.50 \times 3\tfrac{1}{3} = \$55$$

Note: See if you can work all of the foregoing interest Illustrative Examples without looking at the book solutions.

COMPOUND INTEREST

Banks calculate interest either semi-annually (every six months) or annually. If the interest is not collected, that is, if it is allowed to

remain in the bank, the unpaid interest is added to the principal and this is called the *amount.* Interest for the following period is calculated on this *amount.* This is called Compound Interest.

ILLUSTRATIVE EXAMPLE

Find the *amount* of \$500 at 6% for 4 years, compounded annually.

\$500	(principal)
.06	(interest rate)
\$30.00	(interest at end of 1st year)

Interest for the second year will be calculated on the *amount* at the end of the first year. The next step, then, is to find this *amount.*

\$500	(principal)
30	(interest for 1st year)
\$530	(*amount* at end of 1st year)

Interest for second year will be

\$530	(*amount* at end of 1st year)
.06	(interest rate)
\$31.80	(interest at end of 2nd year)

The *amount* at the end of the second year is

\$530.00	(*amount* at end of 1st year)
31.80	(interest for 2nd year)
\$561.80	(*amount* at end of 2nd year)

Interest for the third year is based on the *amount* at the end of the second year and will be

\$561.80	(*amount* at end of 2nd year)
.06	(interest rate)
\$33.7080	(interest at end of 3rd year)

As the last two figures are more than half a cent, the interest is \$33.71.

The *amount* at the end of the third year (must be found as a basis for interest for the fourth year) is

\$561.80	(*amount* at end of 2nd year)
33.71	(interest for 3rd year)
\$595.51	(*amount* at end of 3rd year)

Interest for the fourth year will be

$$\begin{array}{rl} \$595.51 & (\textit{amount} \text{ at end of 3rd year}) \\ .06 & (\text{interest rate}) \\ \hline .7306 & (\text{interest for 4th year}) \end{array}$$

Here the last two figures are less than half a cent, so they may be dropped. The interest is then \$35.73.

The example calls for the *amount* at the end of 4 years and this will be

$$\begin{array}{rl} \$595.51 & (\textit{amount} \text{ at end of 3rd year}) \\ 35.73 & (\text{interest for 4th year}) \\ \hline \$631.24 & (\textit{amount} \text{ at end of 4th year}) \end{array}$$

The answer is \$631.24.

To compute compound interest semi-annually (every six months) the interest is calculated for one-half year at the given rate. This unpaid interest is added to the principal to make the *amount* at the end of six months. Interest for the next period of six months is calculated based on the *amount* so found. The process is exactly as shown for interest compounded annually except that the *amount* must be calculated at the end of every six-month period as a basis for the interest for the next half year.

PRACTICE PROBLEMS

After you have worked all of the following problems compare your answers with the correct answers shown on page 49.

Be sure to work every problem because you gain experience and skill by actually working problems. If you have any trouble with the problems, review Lessons 1, 2, and 3 carefully. These lessons contain all the information you need to work the problems.

1. Find the interest on \$500 at 6% for 1 year.
2. Find the interest on \$800 at 4% for 1 year.
3. Find the interest on \$1,500 at 5% for 2 years.
4. Find the interest on \$347 at 3% for 3 years.
5. Find the interest on \$470 at 6% for 6 months.
6. Find the interest on \$1,200 at 3% for 8 months.
7. Find the interest on \$900 at $4\frac{1}{2}$% for 3 months.
8. At what rate of interest will \$1,800 yield \$225 interest in $2\frac{1}{2}$ years?
9. At what rate of interest will \$5,000 yield \$1,100 interest in $5\frac{1}{2}$ years?
10. At what rate of interest will \$2,150 yield \$150.50 interest in 2 years?

11. At what rate of interest will \$750 yield \$125 interest in 8 years + 4 months?

12. At what rate of interest will \$4,050 yield \$1,336.50 interest in 6 years?

13. What principal will yield \$15 interest in 6 months at 3%?

14. What principal will yield \$96 interest in 3 years at 4%?

15. What principal will yield \$2,028 interest in 4 years at 6%?

16. What principal will yield \$195.75 interest in 3 years at $4\frac{1}{2}$%?

17. In what time will \$500 gain \$56.25 interest at $4\frac{1}{2}$%?

18. In what time will \$6,000 gain \$1,680 interest at 4%?

19. In what time will \$25,000 gain \$4,125 interest at $5\frac{1}{2}$%?

20. In what time will \$1,800 gain \$504 interest at 3%?

21. Find the *amount* when \$300 is loaned for 3 years at 6%.

22. Find the *amount* when \$850 is loaned for $6\frac{1}{2}$ years at 4%.

23. A man borrowed \$9,000 to build a house. The bank charged him interest at the rate of 4%. If he paid back the loan in 10 years what was the total interest?

24. If you borrow \$950 for 2 years at 5% interest what is the *amount?*

25. A gasoline station owner borrowed \$5,500 to rebuild his station. He promised to pay the loan in 3 years with interest at 6%. What was the *amount?*

26. If you lend \$400 at 4% interest, how much interest would you receive in $1\frac{1}{2}$ years?

27. Suppose you bought \$2,000 worth of bonds which paid 6% interest every year. How much money in interest would you make in 5 years?

28. Suppose you bought a new auto which cost \$1,200 and that you paid a down payment of \$400. If you then signed an agreement to pay the balance at the end of 18 months with interest at 6%, what total amount would you have to pay?

29. A man borrowed \$480, agreeing to pay interest at 5% per year. If he paid the loan in 14 months, how much was the *amount?*

30. A man repaid a loan at the end of $6\frac{1}{2}$ years. What was the principal if he had paid a total interest of \$270.40 at the rate of 5%?

Lesson 4

For Step 1, recall what you have learned or read about business transactions involving discounts and paying of commissions. For Step 2, learn the different ways in which these may be applied. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

DISCOUNT

In store windows you have often seen signs something like these:

Clothing Sale	*Furniture*
30% off	*at 25% to 50% reductions*

Such signs mean that the merchants are offering their merchandise at a price 30%, 25%, or 50% less than the regular price.

Suppose a clothing merchant regularly asks $50 for suits of clothes. During a sale he may advertise that he will sell at 30% discount. This means he is offering 30% off on a $50 suit. Thus 30% of $50 = $50 × .30 = $15; and $50 − $15 = $35, the "sale" or discount price. In other words a **discount** of 30% means the original price **minus** 30% of it.

In many lines of business it is a common practice to give a discount when goods purchased are paid for within a certain period. For example, a merchant may buy $500 worth of goods. On the bill he received a statement may appear saying that a 3% discount will be allowed if the bill is paid within 10 days. To find the amount of the bill less 3% discount, it is necessary to find 3% of $500 and subtract that from the $500. Thus $500 × .03 = $15 and $500 − $15 = $485. The $485 is the amount the merchant will pay if he pays within 10 days.

The following definitions explain the names used in discounts.

Marked Price. In stores, in catalogues, etc., the regular prices are called the marked prices. This is the price at which the goods were originally intended to sell.

Discount. The amount of money saved, which is the amount taken off the regular price, is called the discount.

Net Price. The amount remaining after the discount has been subtracted from the marked price is called the net price.

To Find the Discount. Should you want to figure the discount, it can be done by following **Rule 2,** but for convenience **Rule 9A** is given using the new terms you have learned for Discounts.

Rule 9A. *To find the discount, multiply the marked price by the rate of discount.*

This is done (as for **Rule 2**) by changing the discount rate to decimal hundredths, multiplying by the marked price, and pointing off (in the product) as many decimal places as there are in the marked price and in the decimal hundredths together.

To Find the Net Price. If you wish to find the net price when you know the discount and the marked price, use **Rule 9B.**

Rule 9B. *To find the net price, subtract the discount from the marked price.*

The following Illustrative Examples explain the use of both **Rule** 9A and **Rule 9B.**

ILLUSTRATIVE EXAMPLES

1. A dress marked \$40 is sold at a discount of 20%. Find the discount and net price.

Solution

To find the discount use **Rule 9A.**

$$\$40\times 20\% = \$40\times .20 = \$8 \text{ (discount)}$$

To find the net price use **Rule 9B.**

$$\$40 - \$8 = \$32 \text{ (net price)}$$

2. A merchant bought \$350 worth of goods. The bill he received was marked, "3% discount if paid within 10 days." What was the discount and net price to the merchant if he paid the bill within the 10-day period?

Solution. To find the discount use **Rule 9A.**

$$\$350\times 3\% = \$350\times .03 = \$10.50 \text{ (discount)}$$

To find the net price use **Rule 9B.**

$$\$350 - \$10.50 = \$339.50 \text{ (net price)}$$

To Find Per Cent of Discount. Sometimes you want to find the *per cent of discount* (or the discount rate) when the marked price and discount are known. This is done by the same method used in **Rule 3,** but, for convenience, **Rule 10A** is given using the special terms that apply to discount.

Rule 10A. *To find the per cent of discount, divide the discount by the marked price.*

ILLUSTRATIVE EXAMPLE

1. A dress marked \$40 is sold at a discount of \$10. Find the per cent of discount.

Solution. Use **Rule 10A.**

$$40)\overline{10.00}(.25 = 25\% \text{ (discount)}$$
$$\begin{array}{r} 8\,0 \\ \hline 2\,00 \\ 2\,00 \\ \hline \end{array}$$

Proof

$$\$40\times .25 = \$10$$

To Find the Marked Price. When it is necessary to find the marked price, it is done by the same method used for **Rule 4.** For convenience, **Rule 10B** is given. It is the same as **Rule 4,** but uses the terms that apply to discount.

Rule 10B. *To find the marked price, divide the discount by the rate of discount and multiply the quotient by 100.*

ILLUSTRATIVE EXAMPLE

1. If the discount on a dress is $16 and the per cent of discount is 40%, find the marked price.

Solution. Use **Rule 10B.**

$$40)16.0(.4$$
$$\underline{16\ 0}$$

$$.4 \times 100 = 40 \text{ (marked price)}$$

Proof $\$40 \times 40\% = \$40 \times .40 = \$16$

Another common practice in business is to give one, two, or three discounts on prices given in catalogues, for example. Most catalogues cannot be reprinted as often as prices change, so the prices are printed at a given figure (marked prices) and separate discount lists are sent out to care for price changes. The remainder, after all discounts are deducted is called the **net price.**

From this point on, no rules are given because you can see easily that the solutions of discount and other business transactions are based on the percentage operations you are already familiar with.

ILLUSTRATIVE EXAMPLES

1. Find the net price of a bill of goods for $75.40, discounts of 20% and 10% being allowed.

Solution

Instruction	*Operation*
Step 1	**Step 1**
The first discount is 20%. We know 20% = .20. Then follow **Rule 9A.** This is the same as **Rule 2** where you learned to find a per cent of a number.	$75.40 (marked price) .20 (per cent of first discount) 0000 15080 $15.08ØØ = $15.08 (discount)

Step 2

The \$15.08 is subtracted from \$75.40 leaving a *first* remainder of \$60.32.

$$\begin{array}{rl} \$75.40 & \text{(marked price)} \\ \underline{15.08} & \text{(discount)} \\ \$60.32 & \text{(first remainder)} \end{array}$$

Step 3

The second discount is 10% = .10. To find the second discount we find 10% of the first remainder.

$$\begin{array}{rl} \$60.32 & \text{(first remainder)} \\ \underline{.10} & \text{(second discount)} \\ 0000 & \\ 6032 & \\ \$6.0320 = \$6.03 & \text{(second discount)} \end{array}$$

Step 4

The *second* remainder is found by subtracting the second discount from the first remainder.

$$\begin{array}{rl} \$60.32 & \text{(first remainder)} \\ \underline{6.03} & \text{(second discount)} \\ \$54.29 & \text{(second remainder)} \end{array}$$

Step 5

After discounts of 20% and 10% have been taken, the net price is thus \$54.29.

\$54.29 (net price)

2. What is the difference on a bill of \$650 between a single discount of 30% and discounts of 25% and 5%?

Solution

Instruction / *Operation*

Step 1

The single discount of 30% is found as explained in **Rule 9A.**

$$\begin{array}{rl} \$650 & \text{(marked price)} \\ \underline{.30} & \text{(per cent of discount)} \\ 000 & \\ \underline{1950} & \\ \$195.00 = \$195 & \text{(discount)} \end{array}$$

The \$195 is the discount when a single discount of 30% is allowed. Next we will find the two discounts at 25% and 5% by the same method that was used in Example 1.

Step 2

Find the discount at 25%.

$650	(marked price)
.25	(per cent of discount)
3250	
1300	
$162.50	(first discount)

Step 3

The first remainder (as in Example 1) is found by subtracting the first discount from the marked price.

$650	(marked price)
162.50	(first discount)
$487.50	(first remainder)

Step 4

The second discount of 5% is found by finding 5% of the first remainder. The $24.375 can be called $24.38 as explained in the "Summary" in Lesson 1.

$487.50	(first remainder)
.05	(per cent of discount)
$24.3750 = $24.38	(second discount)

Step 5

Add the discounts obtained from the 25% and 5%. The $186.88 is the sum of the two discounts.

$162.50	(first discount)
24.38	(second discount)
$186.88	(total discount)

Step 6

The difference between a single discount of 30% and two discounts of 25% and 5% is therefore $8.12.

$195.00	(30% discount)
186.88	(25% and 5% discounts)
$ 8.12	(difference)

Note that the single discount of 30% amounts to $8.12 more than the two discounts of 25% and 5%.

COMMISSIONS

A commission is generally paid to a person who is engaged in buying or selling goods for another person. There are several different ways in which such business is carried on. The following are typical examples.

A **salesman** is a person who sells goods for a manufacturer or other producer. For example, automobile manufacturers have salesmen all over the country selling their cars. The salesmen are paid a certain per cent of the selling price of the cars they sell.

An **agent** generally sells such things as insurance. Or, sometimes merchants have agents who buy goods for them. In either case the agent is paid a certain per cent of the price of all the items he sells or buys.

A **commission man,** in the most typical cases, buys and sells in a little different manner from salesmen or agents. Farmers send their products to commission men and instruct them to sell these products when the price is right or for the best price they can get. Cattle raisers and fruit growers also use commission men. Sometimes commission men buy farm products instead of selling them. In any event the commission man is paid a certain per cent of the price of the products.

The following definitions explain the terms used.

Commission. The money paid to salesmen, agents, and commission men is called commission. As explained previously, commission is a certain per cent of selling price. Thus if you, as a salesman, sold a $1,200 automobile and your commission was 10%, you would be paid 10% of $1,200 or $120. Or, if you were a commission man and sold a carload of potatoes for a farmer, you would be paid a certain per cent of the total amount you received for the potatoes.

Rate of Commission. If a salesman receives 10% of the selling price of a car, then 10% is the rate of commission.

Net Proceeds. Suppose a salesman sells $1,000 worth of goods. His commission figured at a rate of 10% would be $100. The net proceeds is then $1,000−$100 or $900. In other words, the amount left after the commission is deducted is called the net proceeds.

ILLUSTRATIVE EXAMPLES

1. Find the net proceeds resulting from a sale which totals $500 for which a rate of commission of 5% is paid.

Solution. First find the commission. The rate of commission (5%) is figured on the amount of the sale ($500).

Thus 5% of $500 = $500 × .05 = $25 (commission). You will note that we used **Rule 2** because commission is a percentage.

The net proceeds is the amount of the sale minus the commission.

$$\$500-\$25=\$475 \text{ (net proceeds)}$$

2. An agent whose rate of commission is 5% remits \$3,800 to a merchant *after* deducting his commission. Find the total amount of the sale.

Solution. The agent's commission is figured as 5% of the total amount of the sale. You know that the \$3,800 is the money left after the agent deducted his 5% commission. Thus, it is easy to see that if the agent deducted 5% he sent the merchant only 95% because 100% is the total sale and the agent deducted 5%. Therefore the \$3,800 is really 95%.

Here we have a given amount of \$3,800 and the per cent (95%) and we want to find what amount \$3,800 is 95% of. (See **Rule 4.**) Following **Rule 4,** we find what 1 per cent of the unknown number is. We do this by dividing the given number by the per cent we have just found by reasoning (95%).

$$\$3800 \div 95=\$40$$

$$\$40 \times 100=\$4000 \text{ (total amount of sale)}$$

Proof

$$\$4000 \times .95=\$3800.$$

3. A merchant sent his commission man \$1,260. This sum was to pay the cost of the merchandise and the commission of 5%. What was the sum invested in the merchandise?

Solution. The commission man must figure his commission on the amount of money he actually pays for the merchandise. The \$1,260 includes the money paid for merchandise and the commission. In other words the \$1,260 is price of merchandise (100%) plus commission (5%). Thus \$1,260 is really 105% of amount invested in merchandise. Here we can use **Rule 4** again because we want to find what number \$1,260 is 105% of.

$$\$1260 \div 105=\$12$$

$$\$12 \times 100=\$1200 \text{ (sum invested)}$$

Proof

$$\$1200 \times 105\%=1200 \times 1.05=\$1260$$

or, $\$1200 \times .05\%=\$60$ commission.

And $\$1260-60=\$1200.$

Lesson 5

For Step 1, recall what you already have learned in percentage. For Step 2, apply that knowledge on new practical business procedure. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

TAXES, DUTIES, CUSTOMS

Taxes are paid on real estate, incomes, personal property, amusements, automobiles, safety deposit boxes, telegrams, tobacco, etc. In most cases taxes are a means of raising money to pay the expenses of city, county, state, and federal governments, highways, etc.

The taxes we pay are figured as a certain per cent of the selling prices, value of property, etc. Sometimes taxes are in terms of 1%, 2%, 4%, etc. In other cases, such as for real estate, the tax is reckoned in mills. (A mill is $\frac{1}{10}$ of a cent.) A tax of 25 mills on a dollar valuation of property would be the same as $2\frac{1}{2}$ cents on a dollar or $2\frac{1}{2}$%.

Duties are levied on goods coming into our country from foreign countries. Sometimes the duty is reckoned on the value of incoming goods. In such cases the duty is called *ad valorem duty.* In other cases duty is figured according to weight and is then called *specific duty.*

ILLUSTRATIVE EXAMPLES

1. What is the duty on $15,000 worth of furs if the *ad valorem* duty is 40%?

Solution. The *ad valorem* duty is figured according to *value.* Here the value is $15,000. The duty is 40% of $15,000. (See **Rule 2.**)

$$\begin{array}{r} \$15000 \\ .40 \\ \hline 00000 \\ 60000 \\ \$6000.00 \end{array}$$

The duty is $6,000.

You will note that this problem, like all others you have studied, is based on percentage and that the basic rules you studied in Lessons 1 and 2 still apply.

2. The specific duty on grape sugar is $1\frac{1}{8}$¢ a pound. If the duty is \$135 how many pounds were imported?

Solution. Specific duty is based on weight. Before we can solve this problem, like all problems involving taxes or duties, the $1\frac{1}{8}$¢ must be changed to terms of dollars. To start with, $1\frac{1}{8}=1.125$. (This is explained in Section 5.) Now, we know that 5 cents is .05 of a dollar because it is the same as $\frac{5}{100}=.05=\$.05$. In like manner $1.125¢=\frac{1.125}{100}=.01125=\$.01125$.

If \$135 is the total duty paid, and if \$.01125 is the duty per pound, then $\$135\div.01125=12000$ pounds.

```
.01125)135.00000(12000
       112 5
        22 50
        22 50
           000
```

The above division follows the rule given in Section 5 relative to dividing decimals.

Answer = 12,000 pounds.

Proof

$$12000\times\$.01125=\$135.$$

3. The assessed value of a man's property is \$2500, and the tax rate is 2 mills. Find the amount of his taxes.

Solution. You know that a mill is $\frac{1}{10}$¢ or .1 cent. Then 2 mills = .2 cent. If the tax is 2 mills (.2 cent) per \$1, then for \$2,500 it would be $.2\times2{,}500=500$ cents or \$5.00 (Answer).

INSURANCE

There are many forms of insurance such as life, accident, fire, tornado, auto, hail, theft, etc. All forms are based on the same general principle of protecting people against misfortune. Here, as a typical example, we will think in terms of fire insurance. The following definitions explain the important words used in connection with insurance.

Premium. This is the amount of money paid to an insurance company at regular intervals for the protection provided.

Policy. When you take out insurance with an insurance company, you are given a written contract which outlines the extent

of coverage, the amount for which you are insured, etc. This contract is called a policy.

Policyholder. The insured, or person having the policy, is called the policyholder.

Face of Policy. This is the coverage on which the policyholder pays his premium.

Rates. When you buy fire insurance you may pay, say, 20¢ per \$100 worth of insurance. (Fire insurance rates are generally quoted as so much per \$100.) This 20¢ is called the rate.

Most fire insurance policies are written for a term of more than one year, a three-year period being most common. In such cases the rate charged is found by adding 75% of the annual rate (yearly rate) per \$100 for each additional year. Thus the rate for two years is $1\frac{3}{4}$ times the annual rate, while for three years it is $2\frac{1}{2}$ times the annual rate and so on. For example, if the rate per hundred for one year is 20¢, the rate per hundred for two years would be 35¢, and the rate per hundred for three years would be 50¢.

ILLUSTRATIVE EXAMPLES

1. A man insured his house for \$8000, for one year, at 20¢ per \$100. What premium did he pay?

Solution. Face of policy = \$8,000. Rate = 20¢ per \$100. An easy way to solve this problem is first to find how many times \$100 the face of the policy equals and multiply by 20¢.

$\frac{\$8000}{\$100} = 80$. This means the man is paying for 80 times \$100 worth of insurance.

$$80 \times 20¢ = 80 \times .20 = \$16 \text{ (premium)}$$

Alternate Solution. A rate such as 20¢ per hundred can easily be expressed in *per cent* form too. Notice that a rate of \$1 per \$100 would be 1% of the face of the policy. Thus 20¢ per \$100 represents a rate of $\frac{1}{5}$ of 1% because $.20 = \frac{1}{5}$. Then, in the above example, 1% of \$8,000 is \$80 and $\frac{1}{5}$ of 1% of \$8,000 is $\frac{1}{5}$ of \$80 or exactly \$16.

2. Suppose you owned a house the actual value of which was \$10,000. If you insured the house for $\frac{3}{5}$ of its value at a rate of 16¢ per \$100, what would your premium be for 3 years?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Find the actual amount you are going to insure for first. This is done using the rules of Section 4 to multiply a fraction by a whole number.	$\frac{3}{5}$ of $10,000= $\frac{3}{5}\times 10000=\$6,000$
Step 2	**Step 2**
Find the number of times $100 worth of insurance you are paying for.	$\$6,000 \div \$100 = 60$ You are paying for $60\times\$100$ worth of insurance.
Step 3	**Step 3**
Find the rate, remembering that for three years the rate is $2\frac{1}{2}$ times the yearly rate. The yearly or annual rate is 16¢ per $100.	$16¢\times 2\frac{1}{2}=16\times 2.5=40¢$ The rate is thus 40¢ on a three-year term.
Step 4	**Step 4**
Find the premium. We know you would be paying for 60 times $100 worth of insurance. Thus we simply multiply 60 by 40¢.	$\begin{array}{r} 60 \\ \underline{.40} \\ 00 \\ 240 \end{array}$ $40¢=\$.40$ $\$24.00=\$24$ (Answer)

PRACTICE PROBLEMS

The following problems cover all of Lessons 4 and 5.

If you have any trouble with one or more of the problems it is because you have not mastered the lessons. In such case review these lessons carefully. Every principle to be applied in solving the problems is explained in the lessons.

After you have worked all of the following problems, turn to page 49 and compare your answers with the answers given.

Be sure to work every problem because you *learn by doing* and greatly increase your skill.

Discount Problems

1. Merchandise is bought to the amount of $1,250. The merchant is offered a $3\frac{1}{2}\%$ discount for cash. What did the goods cost him if he took advantage of the discount?

2. Which is the more profitable, and by how much, for a buyer to choose on a \$3,500 purchase, a single discount of 15% or discounts of 10% and 5%?

3. I obtained a discount of \$312.20 on a purchase of \$8,920. What rate of discount was I allowed?

Commission Problems

4. What is the value of stock that can be bought for \$9,682, allowing the broker a commission of 3%?

5. An agent in Chicago purchased 4,700 bushels of wheat at 75 cents a bushel. What was his commission at $1\frac{1}{2}$% of the purchase money?

6. A real estate agent sold a house for a certain sum of money and remitted \$19,600 to the owner after deducting his commission of 2%. For how much did the agent sell the house?

7. What commission must be paid for collecting \$17,380 at $3\frac{1}{2}$%?

Taxes and Duties Problems

8. Mr. A's property is valued at \$5,000 and his tax rate is 13 mills. What is the amount of his taxes?

9. A merchant imported 500 boxes of cigars (each weighing 1 pound and containing 100 cigars) invoiced at \$3.50 a box. The duty was 25% *ad valorem* and \$4.50 a pound. What was the total duty paid?

10. The *ad valorem* duty on 800 yards of carpet valued at \$1.65 a yard is \$330. What was the rate of duty?

11. A farmer owns 90 acres of land worth \$100 an acre. This was assessed for taxation at $\frac{2}{3}$ of its value. The rate was $20\frac{1}{2}$ mills. What taxes did he pay?

12. The specific duty on sugar is $3\frac{1}{4}$ cents per pound. What would be the amount of duty on a shipment of 52 tons? (1 ton weighs 2000 pounds.)

Insurance Problems

13. A house valued at \$20,000 was insured for $\frac{4}{5}$ of its value, at a yearly rate of 24¢ per \$100. What is the yearly premium?

14. How much money could be saved in Problem 13 on the basis of a 3-year premium?

15. If you insure your house for \$9,500, at a yearly rate of 24¢ per \$100, and your furniture for \$2,400, at a yearly rate of 28¢ per \$100, how much total premium must you pay for one year?

Miscellaneous Problems

16. If the marked price of an article is \$60 and the discount is \$20 what is the per cent of discount?

17. If the marked price of an item is \$50 and a sale price is \$37, what is (a) the discount, (b) the per cent of discount?

18. Suppose you, as a salesman, sold \$1,400 worth of goods. If your rate of commission was 10%, find the net proceeds.

19. What is the cost to insure merchandise valued at \$9,500 for a year if the rate is 66¢ per \$100?

20. Find the premium for insuring a factory for \$15,000 for two years if the rate is 28¢ per \$100.

ANSWERS TO PRACTICE PROBLEMS

Lesson 1, Page 11

1. 7. **2.** 24.05. **3.** 3.564. **4.** 3.975 **5.** 33.75. **6.** 390.00. **7.** 330.00. **8.** 255. **9.** 135. **10.** 464.64. **11.** 443.75. **12.** 11.745. **13.** 1400. **14.** 47. **15.** \$9,200. **16.** \$2600.

Lesson 1, Page 14

1. \$2.00. **2.** 60 bushels. **3.** 39 feet. **4.** 181 acres. **5.** 301 rods. **6.** 62 gallons. **7.** 90 miles. **8.** \$361.50. **9.** 216. **10.** 935. **11.** 224. **12.** 75. **13.** 100. **14.** 48. **15.** 595.

Solutions to Review Problems, Lesson 1, Page 14

1. The term *hundred* is used with whole numbers (not decimals) such as 100 (one hundred), 200 (two hundred), 300 (three hundred), etc. This term is never used with a number less than 100 in value.

The term *hundredths* is used with decimal numbers such as .10 (ten hundredths), .20 (twenty hundredths), .40 (forty hundredths), etc. This term is used only with numbers which are a fractional part of 100.

2. Another name for this is "per cent." You learned that per cent means the number of hundredths one has in mind. For example, .20 is read twenty hundredths. It can also be written $\frac{20}{100}$. In either case it indicates the number of hundredths.

3. You learned that any per cent over 100 is more than a whole. The 250% expressed as a decimal is 2.50. To find 250% of 350 we multiply 350 by 2.50.

$$\begin{array}{r} 350 \\ 2.50 \\ 000 \\ 1750 \\ \underline{700} \\ 875.00 \end{array}$$

Answer is 875.

4. (a) From Table of Equivalents the equivalents for $37\frac{1}{2}\%$ are $\frac{3}{8}$, $.37\frac{1}{2}$, and $\dfrac{37\frac{1}{2}}{100}$.

(b) The Table does not show $13\frac{1}{3}\%$ so we must calculate.

$$13\tfrac{1}{3} = \tfrac{40}{3}$$

$$\tfrac{40}{3} \div 100 = \frac{\overset{\overset{2}{\cancel{4}}}{\cancel{40}}}{\underset{\underset{15}{\cancel{30}}}{\cancel{300}}} = \tfrac{2}{15}$$

The equivalents are therefore

$$\tfrac{2}{15},\ .13\tfrac{1}{3},\ \text{and}\ \frac{13\frac{1}{3}}{100}$$

5. The 8.6% must be changed to a decimal hundredth. We know that 8.6 is the same as $8\frac{6}{10}$ and that $8\frac{6}{10}$ expressed as a fractional hundredth is $\frac{8\frac{6}{10}}{100}$. This means $8\frac{6}{10}\div 100$. Change $8\frac{6}{10}$ to an improper fraction; divide by 100.

$$8\tfrac{6}{10}=\tfrac{86}{10}$$

$$\frac{86}{10}\div 100=\frac{43}{500}=\frac{43}{500}$$

Then $\frac{43}{500}=43\div 500=.086$.

Also we could have reasoned that 8.6% is the same as $\frac{8.6}{100}$ and that $8.6\div 100=.086$. Then $.086\times 256=22.016$.

6. (a) 9% means the same as $\frac{9}{100}$ because per cent indicates the number of hundredths. The $\frac{9}{100}$ indicates that 9 is to be divided by 100. Following **Rule 1**, $9\div 100=.09$. Thus $9\%=.09$ (decimal hundredths). Another way of changing 9% to decimal hundredths is to remember that *hundredths* must have two places to right of decimal point. Thus to write 9% as a decimal we know we must put a zero in front of the 9 making it .09 (two places to right of decimal point). We cannot put the zero after that 9 or we would have .90 which would make 90 hundredths instead of the 9 hundredths we should have.

(b) 6.7% means the number of hundredths, so it is the same as $\frac{6.7}{100}$. Remember that the numerator of a fraction shows the parts we are thinking of. Here we are thinking of 6.7 hundredths so the 6.7 becomes the numerator. The denominator is always 100, because in percentage we think of things being divided into 100 parts (hundredths). Dividing 6.7 by 100 (**Rule 1**) gives .067.

(c) 12% means the same as $\frac{12}{100}$. Using **Rule 1** we see that $12\div 100=.12$. Also we can always remember that where a per cent is composed of two digits (See Section 1) we can change it to decimal hundredths merely by putting a decimal point in front of it.

7. 16.7% means the same as $\frac{16.7}{100}$. By **Rule 1**, $16.7\div 100=.167$. Remember that any per cent such as 16.7, 87.2, 14.9, 23.6, etc., can be changed to decimal hundredths by moving the decimal point two places to the left.

8. The 150% means the same as $\frac{100}{100}+\frac{50}{100}$. In percentage a whole unit is 100. Thus the 150% is 1 whole unit plus $\frac{50}{100}$ of another whole unit. Using **Rule 1** the 150% becomes $150\div 100=1.50$.

The 1.5% means $\frac{1}{100}$ plus $\frac{.5}{100}$. The $\frac{.5}{100}$ is just half of $\frac{1}{100}$. So the difference between 150% and 1.5% is the difference between $(\frac{100}{100}+\frac{50}{100})$ and $(\frac{1}{100}+\frac{.5}{100})$ or the difference between 1.50 and .015. If we find 150% and 1.5% of 240 the difference is easy to see.

$150\%=1.50$	$1.5\%=.015$
240	240
1.50	.015
000	1200
1200	240
240	$3.600=3.6$
360.00	

150% of $240=360$ while 1.5% of $240=3.6$

9. The $\frac{80}{100}$ is in fractional hundredths (see Table of Equivalents). We learned that to change fractional hundredths to a proper fraction we simply reduce it to lowest terms.

$$\frac{\overset{4}{\cancel{80}}}{\underset{5}{\cancel{100}}}=\frac{4}{5}$$

If this is not clear to you, refer back to Section 3 and review the process of reducing fractions to lowest terms.

10. The first method is as follows

$$\begin{array}{r} 366 \\ \\ 122 \\ 1098 \\ 2928 \\ \hline 305.00 \end{array}$$

$\frac{1}{3}$ of $366=\frac{1}{\cancel{3}}\times\overset{122}{\cancel{366}}=\frac{1\times122}{1}=122.$

$305.00=305$ (Answer)

The second way (short method) is as follows.

$83\frac{1}{3}\%=\frac{5}{6}$ (proper fraction equivalent)

$$\begin{array}{r} 6\overline{)366} \\ 61 \\ 5 \\ \hline 305 \end{array}$$ (Answer)

Lesson 2, Page 18

1. 15 per cent. **2.** 40 per cent. **3.** 50 per cent. **4.** 25 per cent. **5.** $66\frac{2}{3}$ per cent. **6.** $8\frac{1}{3}$ per cent. **7.** 15 per cent. **8.** 75 per cent. **9.** 40 per cent. **10.** 20 per cent.

Lesson 2, Page 20

1. 40 per cent. **2.** 90 per cent. **3.** 60 per cent. **4.** 30 per cent. **5.** 70 per cent. **6.** 80 per cent.

Lesson 2, Page 22

1. 10. **2.** 400. **3.** 400. **4.** 1,900. **5.** 300. **6.** 800. **7.** 3,200. **8.** 10,000.

Answers to Review Problems, Lesson 2, Page 24

1. Use **Rule 3.**

$=1680\div3500=.48=48\%$. Ans.

2. Use **Rule 2.** $3500\times.25=875$ (Answer)

3. Use **Rule 2.** $40\times.90=36$ (Answer)

4. Use **Rule 4.**

$$\begin{array}{r} 40)32.0(.8 \\ 32\ 0 \\ \hline \end{array}$$

$.8\times100=80$ miles

5. Use **Rule 4.**

$$\tfrac{10}{40}=10\div 40=.25 \text{ or } 25\% \text{ (Answer)}$$

6. Use **Rule 4.**

$$\begin{array}{l} \underline{45)}1125\underline{(25} \\ \quad\;\; \underline{90} \\ \quad\;\; 225 \\ \quad\;\; 225 \end{array}$$

$$25\times 100=\$2500 \text{ (Answer)}$$

7. Use **Rule 3.**

$$\tfrac{2500}{3000}=\tfrac{5}{6}=83\tfrac{1}{3}\%$$

Here we made use of the Table of Equivalents.

8. Use **Rule 2.**

$$\$20\times .30=\$6 \text{ (Answer)}$$

9. Use **Rule 4.**

$$\begin{array}{l} \underline{15)}240\underline{(16} \\ \quad\;\; \underline{15} \\ \quad\;\;\; 90 \\ \quad\;\;\; 90 \end{array}$$

$$16\times 100=\$1600 \text{ (Answer)}$$

10. 3,000.
11. 1,200.
12. 800.

Lesson 3, Page 32

1. \$30. **2.** \$32. **3.** \$150. **4.** \$31.23. **5.** \$14.10. **6.** \$24. **7.** \$10.12½, call it \$10.13. **8.** 5%. **9.** 4%. **10.** 3½%. **11.** 2%. **12.** 5½%. **13.** \$1,000. **14.** \$800. **15.** \$8,450. **16.** \$1,450. **17.** 2½ years. **18.** 7 years. **19.** 3 years. **20.** 9⅓ years. **21.** \$354. **22.** \$1,071. **23.** \$3,600. **24.** \$1,045. **25.** \$6,490. **26.** \$24. **27.** \$600. **28.** \$872. **29.** \$508. **30.** \$832.

Lessons 4 and 5, Page 44

1. \$1,206.25. **2.** Single discount. \$17.50. **3.** 3½%. **4.** \$9,400. **5.** \$52.875 or \$52.88. **6.** \$20,000. **7.** \$608.30. **8.** \$65.00. **9.** \$2,687.50. **10.** 25%. **11.** \$123.00. **12.** \$3,380. **13.** \$38.40. **14.** \$19.20. **15.** \$29.52. **16.** 33⅓%. **17.** (a) \$13; (b) 26%. **18.** \$1,260. **19.** \$62.70. **20.** \$73.50.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or final examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on page 52.

If you miss more than two of the problems it means you should review the whole book carefully.

Do not try this trial examination until you have worked every practice problem in the book.

Do not start the final examination until you have completed this one.

1. Change each of the following fractions to per cent.

(a) $\frac{4}{5}$ (b) $\frac{5}{7}$

2. Using the *short method* find

(a) $18\frac{3}{4}\%$ of 144 (b) $71\frac{3}{7}\%$ of 182

3. If you worked 20 problems correctly out of an examination consisting of 25 problems, what per cent were correct?

4. A store owner insured his \$20,000 stock for $\frac{3}{5}$ of its value at a rate of 24¢ per \$100. He also insured his \$25,000 building for $\frac{4}{5}$ of its value at a rate of 28¢ per \$100. What was his total premium for 2 years?

5. Express 275% as you would use it in finding that per cent of a number.

6. If your salary was \$150 a month and you received a 15% raise in salary, what is your new salary?

7. If you borrowed \$1,500 at 6% interest for $2\frac{1}{2}$ years, what total amount would you have to pay?

8. A boring machine had a speed of 360 revolutions per minute. In order to increase the output, the speed was raised $8\frac{1}{3}\%$. What was the new speed?

9. A man investing \$2,750 in real estate sold his holdings at the end of 3 years for \$9,515.00. What rate of interest did he make on his investment?

10. What is the interest on \$1,250 at 6% for 3 years, 4 months, and 15 days? (Count 15 days as half a month.)

Note

Before you start on the "Final Examination" be sure you have done the following things.

1. Studied the lessons, rules, etc., until you understand Percentage.
2. Worked *all* Illustrative Examples without looking at the solutions.
3. Worked *all* of the Practice Problems.
4. Worked the "Trial Examination."

All of the above items constitute what we can term *proper preparation*. Unless you have obtained the proper preparation you are apt to fall below a passing grade on the final examination. By working the illustrative and practice problems you obtain a good understanding of percentage principles.

We emphasize the above because unless you do follow such suggestions you will find difficulty not only in the final examination but also in advanced sections and in solving the problems you will meet in everyday life.

The experience of thousands of students has proved these suggestions to be worth following no matter what the amount of study time required.

FINAL EXAMINATION

1. A bank receives \$433.55 interest for money it had loaned for 1 year at $5\frac{1}{2}\%$. What was the principal?

2. Change the following fractions to per cent:

(a) $\frac{9}{16}$ (b) $\frac{3}{32}$ (c) $\frac{9}{32}$

3. Change the following per cent values to fractions in their lowest terms.

(a) $43\frac{3}{4}\%$ (b) $15\frac{5}{8}\%$ (c) $87\frac{1}{2}\%$

4. Solve the following problems using the short method.

(a) Find $37\frac{1}{2}\%$ of 4328 (b) Find $16\frac{2}{3}\%$ of 882

5. An electrician bought 800 feet of copper wire. If 15% of it was stolen and if he used 75% of it, how many feet were left?

6. Property worth \$7,500 was insured for $\frac{2}{3}$ of its value at the rate of 30¢ per \$100 for three years. What was the premium?

7. When water freezes it expands 10% in volume. How many cubic inches of ice would 947 cubic inches of water make if frozen?

8. A laborer working 48 hours a week received 35¢ per hour. His week's wages were raised to \$21.60.

(a) What was the increase per hour?

(b) What per cent raise did he get?

9. A merchant had two bills as follows:

1500 kilowatt hours electricity at 10c per kilowatt hour.

1800 cubic feet gas at 8¢ per cubic foot.

The rate of discount for cash on both items was 10%. What total amount did he pay to settle both bills?

10. Find the interest on \$2,550 at $5\frac{1}{2}\%$ for 4 years and 8 mo.

11. An agent purchased 5,000 bushels of corn at 80¢ a bushel What was his commission at 2%?

12. A small generator used for testing purposes ordinarily produced 1,500 volts. A special test requiring 2,500 volts was necessary. What per cent would the voltage have to be raised?

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. (a) From the Table of Equivalents, we can see that $\frac{4}{5}$ is an equivalent of 80%.

$$\text{(b) } \tfrac{5}{7} = 5 \div 7 = .71\tfrac{3}{7} = 71\tfrac{3}{7}\%$$

2. (a) From the Table of Equivalents we know that $18\frac{3}{4}\%$ has an equivalent of $\frac{3}{16}$.
Then $144 \div 16 = 9$ and $9 \times 3 = 27$. Thus the answer is 27.

(b) $71\frac{3}{7}\% = \dfrac{71\frac{3}{7}}{100}$. We can divide $71\frac{3}{7}$ by 100 if we change the mixed number to an improper fraction as explained in Section 4. Thus $71\frac{3}{7} = \frac{500}{7}$. Then

$$\tfrac{500}{7} \div 100 = \frac{\overset{5}{\cancel{500}}}{\underset{7}{\cancel{700}}} = \tfrac{5}{7}.$$ Then $182 \div 7 = 26$ and $26 \times 5 = 130$ (Answer)

3. This problem calls for **Rule 3.** Following this rule we have

$\frac{20}{25}$. Then $20 \div 25 = .80$ or 80%.

4. $\frac{3}{5}$ of \$20,000 = 20,000 ÷ 5 = 4000 and 4000 × 3 = \$12,000 (stock)

$\frac{4}{5}$ of \$25,000 = 25,000 ÷ 5 = 5000 and 5000 × 4 = \$20,000 (building)

\$12,000 ÷ \$100 = 120 (number of hundreds)

\$20,000 ÷ \$100 = 200 (number of hundreds)

The rate of premium for 2 years is $1\frac{3}{4}$ times that for 1 year.

$$1\tfrac{3}{4} \times 24 = \frac{7}{\underset{1}{\cancel{4}}} \times \frac{\overset{6}{\cancel{24}}}{1} = 42¢ \text{ (rate on stock for 2 years)}$$

$$1\tfrac{3}{4} \times 28 = \frac{7}{\underset{1}{\cancel{4}}} \times \frac{\overset{7}{\cancel{28}}}{1} = 49¢ \text{ (rate on building for 2 years)}$$

120 × 42¢ = \$50.40 (premium on stock)

200 × 49¢ = \$98.00 (premium on building)

\$50.40 + 98.00 = \$148.40 (Answer)

5. In percentage 100% is a whole. Therefore 275% is much more than a whole. In fact it would be 100% + 100% + 75%. Or, as we learned in Lesson 1, we can write this as 2.75. (Answer)

6. This problem calls for **Rule 2.**

15% of \$150 = \$150 × .15 = \$22.50 (increase)

\$150 + \$22.50 = \$172.50 (new salary)

7. This problem calls for **Rule 5.**

\$1500 × 6% = \$1500 × .06 = \$90 (interest for 1 year)

Then \$90 × $2\frac{1}{2}$ = \$225 (interest for $2\frac{1}{2}$ years)

\$1500 + \$225 = \$1725 (total amount)

8. To solve this problem we use **Rule 2.** First we must find $8\frac{1}{5}\%$ of 360, and add this product to 360. We know $8\frac{1}{5}\%$ is the same as $.08\frac{1}{5}$. Then 360 $\times .08\frac{1}{5}$ is

$$\begin{array}{r} 360 \\ .08\frac{1}{5} \\ \hline 72 \\ 2880 \\ \hline 29.52 \end{array}$$

Adding 360 and 29.52 we have 389.52 revolutions per minute, which is the new speed.

9. Subtracting \$2,750 from \$9,515 leaves \$6,765, the amount he actually made in 3 years' time. Here we know the principal, interest, and time. It is an easy matter to see that Rule 7 is required.

$\$6,765 \div 3 = \$2,255$ (interest for 1 year)

$\$2,255 \div 2,750 = .82$ or 82% (interest rate)

10. $\$1,250 \times .06 = \$75.00$ (interest for 1 year)

$\$75 \times 3 = \$225.00$ (interest for 3 years)

$\$75 \div 3 = \$25.00$ (interest for $\frac{1}{3}$ year)

$\$75 \div 24 = \$3.125$ (interest for 15 days)

Here 15 days $= \frac{1}{2}$ month or $\frac{1}{24}$ of a year.

$$\begin{array}{r} \$225.00 \\ 25.00 \\ 3.125 \\ \hline \$253.125 \end{array}$$ or \$253.13 (total interest)

PRACTICAL MATHEMATICS

Section 7

Lesson 1

For Step 1, take notice of the introductory paragraphs in the lessons and bear in mind that this topic will enable you to compute measurements in everyday life. For Step 2, learn the method of computing linear measure. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

DENOMINATE NUMBERS

Having finished the texts on Fractions, Decimals, and Percentage, you are now ready to begin the work in which you apply all of the principles you have learned. Since the beginning of time men have been asking the questions: How much? How many? How far? How long? How heavy?

Nearly all nations have put their stamp on certain measurements which will answer these questions. The English Government has one set of measurements and the French Government has another set. Other nations also have different kinds of measurements. The aim of this book is to take up the study of these measurements.

The numerals, such as 2, 3, 4, 5, etc., when used by themselves, are called **abstract numbers.**

When these numbers are used with the name of something or the name of a unit, such as feet, miles, hours, etc., these are called **concrete numbers.**

Concrete numbers in which units are used are called **denominate numbers.**

If more than one unit is involved in the number, then it is called a **compound denominate number.** Examples of the latter are: 6 feet 8 inches; 2 hours 40 minutes; 8 pounds 12 ounces.

The **reduction of denominate numbers** is the process of changing them from one denomination (one kind of unit) to another without changing their value. The reduction may be from a higher to a lower denomination or from a lower to a higher denomination.

MEASURES

There are two systems of measurement which are legalized in the United States—the English System and the Metric System.

The former is in common use in the United States and in England, while the latter is used in all other countries and in our Governmental Departments and for technical purposes.

A **unit of measure** is a standard by which a quantity such as length, area, capacity, or weight is measured. For example, the length of a piece of pipe may be ascertained by applying the yard or the meter measure; the capacity of a barrel by the use of the gallon or the liter measure; the weight of a body by the use of pounds or kilograms.

The different kinds of measurement will be taken up and discussed in regular order. Tables will be given showing the relations of the different units in the same system. There will also be conversion tables for changing from the English System to the Metric System, or the reverse.

MEASURES OF EXTENSION

Extension is that property of a body by virtue of which it occupies space and has one or more of the three dimensions—length, width, and thickness.

A line, for example, has only one dimension. That dimension is length, and its measurement is accomplished by what we call **linear measure.**

A surface or area has two dimensions—length and width—and its measurement is accomplished by **square measure.**

A solid object has three dimensions—length, width, and thickness—and its measurement is obtained by **cubic measure.**

The standard units of extension in the United States are the **yard** and the **meter.** The yard measures 36 inches. The meter measures 39.37 inches. This shows the relation between the two units in the two different systems, the English and the metric.

Linear Measure

The English measure for length or distance, called **Long Measure,** makes use of the yard as its fundamental unit, with such divisions as feet and inches. For instance, a merchant sells cloth by the yard; a person measures his height in feet, and the length of his arm in inches. The larger units, the **rod** and the **mile,** are used when distances to be measured become so great that the small units are not convenient. A complete table of Long Measure is given here to show you the relation between the different units.

The Metric System is founded on the meter as the fundamental unit, and as it is a decimal system, the smaller and larger units are all decimals, divisions or multiples of the meter. For example, the centimeter which is about two and one-half times smaller than the inch (2.54 cm. = 1 inch) is, as its name indicates, one-hundredth of a meter; while the kilometer, which is about six-tenths of a mile, is one thousand meters. The following tables will give the relation between the different units.

TABLE I

English System with Abbreviations

12 inches (in.)	= 1 foot (ft.)
3 ft. or 36 in.	= 1 yard (yd.)
$5\frac{1}{2}$ yds. or $16\frac{1}{2}$ ft.	= 1 rod (rd.)
220 yds. or $\frac{1}{8}$ mi.	= 1 furlong
320 rds.	= 1 mile (mi.)
1760 yds.	= 1 mile (mi.)
5280 ft.	= 1 mile (mi.)

TABLE II

Metric System with Abbreviations

10 millimeters (mm.)	= 1 centimeter (cm.)
10 cm. or 100 mm.	= 1 decimeter (dm.)
10 dm. or 100 cm.	= 1 meter (m.)
10 m. or 1000 cm.	= 1 decameter (Dm.)
10 Dm. or 100 m.	= 1 hectometer (Hm.)
10 Hm. or 1000 m.	= 1 kilometer (Km.)
10 Km. or 10000 m.	= 1 myriameter (Mm.)

TABLE III

Equivalent Table

1 mm.	=.03937 in.
1 cm.	=.3937 in.
1 m.	=39.37 in.=1.0936+yd.
1 km.	=0.621+mi.=1093.6+yd.
1 in.	=2.54+cm.
1 ft.	=30.48+cm.
1 yd.	=.9144+m.
1 mi.	=1.6+km.=1600+m.

TABLE IV

Nautical Measure

(Used at Sea)

1 nautical mile	=1.15+ land (or statute) mi.
1 nautical mile	=6080.27 ft.
3 nautical miles	=1 league
1 fathom	=6 ft.
1 hand	=4 in.

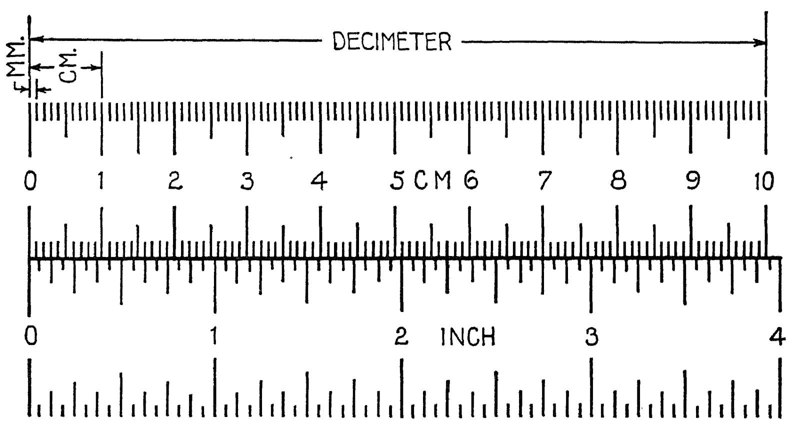


Fig. 1.

Comparison of Scales (Fig. 1). In the Metric System an understanding of the prefixes will be helpful as the same prefixes are used throughout the various metric measures.

Milli—means one-thousandth of unit or $\frac{1}{1000}$ or .001
Centi—means one-hundredth of unit or $\frac{1}{100}$ or .01
Deci—means one-tenth of a unit or $\frac{1}{10}$ or .1
The meter—in this table is the unit base
Deka—means ten times the unit
Hecto—means one hundred times the unit
Kilo—means one thousand times the unit
Myria—means ten thousand times the unit

In Fig. 1, the centimeter scale shows 10 cm. Each of the small divisions is a millimeter. It is easily seen that 10 mm. = 1 cm.

In the inch scale the small divisions are sixteenths of one inch, and 64 sixteenths = 4. So ten centimeters equal nearly four inches. (Refer to the ruler sketch in Book No. 3.)

ILLUSTRATIVE EXAMPLES

1. A man walked $4\frac{1}{2}$ miles an hour for $5\frac{1}{4}$ hours.
   (a) How many rods did he walk?
   (b) How many myriameters did he walk?

Solution

Instruction	*Operation*
Step 1 To find the number of miles walked, multiply $4\frac{1}{2}$ by $5\frac{1}{4}$, reducing fractions to decimals first.	**Step 1** $4\frac{1}{2} = 4.5$ and $5\frac{1}{4} = 5.25$ $4.5 \times 5.25 = 23.625$
Step 2 By Table I, in 1 mile there are 320 rods, so in 23.625 miles there will be $(320 \times 23.625) = 7560$ rd.	**Step 2** $320 \times 23.625 = 7560$ Ans.
Step 3 By Table III, in 1 mile there are 1600 m., so in 23.625 miles there are (1600×23.625) m.	**Step 3** $23.625 \times 1600 = 37800$
Step 4 By Table II, there are 10000 m. in 1 Mm., so in 37800 m. there are $(37800 \div 10000) = 3.78$ Mm.	**Step 4** $37800 \div 10000 = 3.78$ Ans.

2. An athlete ran and jumped 15 feet $7\frac{1}{2}$ inches.
(a) How many centimeters did he jump?
(b) How many meters did he jump?

Solution

Instruction	*Operation*
Step 1 In 1 ft. there are 30.48 cm. In 15 ft. there are (30.48×15) cm.	**Step 1** $30.48\times15=457.20$
Step 2 In 1 in. there are 2.54 cm. In $7\frac{1}{2}$ in. there are $(2.54\times7\frac{1}{2})$ cm., or 2.54×7.5	**Step 2** $2.54\times7.5=19.05$
Step 3 Total number of cm. $=$ $457.20+19.05=476.25$ cm. Ans.	**Step 3** $457.20+19.05=476.25$
Step 4 100 cm. $=1$ m. Then 476.25 cm. $=(476.25\div100)$ $=4.7625$ m. Ans.	**Step 4** $476.25\div100=4.7625$

3. A train was averaging $49\frac{3}{4}$ miles an hour for 3 hours and 20 minutes. How far did it go in myriameters?

Solution

Instruction	*Operation*
Step 1 Reduce 20 min. to fraction of an hour. (60 min. $=1$ hour) In 1 hour, it goes $49\frac{3}{4}$ mi. In $3\frac{1}{3}$ hours, it goes $49\frac{3}{4}\times3\frac{1}{3}=165.83\frac{1}{3}$ mi.	**Step 1** 20 min. $=\frac{20}{60}$ or $\frac{1}{3}$ hour $49\frac{3}{4}=49.75$ $49.75\times3\frac{1}{3}=165.83\frac{1}{3}$
Step 2 By Table III, 1 mi. $=1.6$ km. Then $165.83\frac{1}{3}$ mi. $=$ $(165.83\frac{1}{3}\times1.6)=265.333\frac{1}{3}$ km.	**Step 2** $165.83\frac{1}{3}\times1.6=265.333\frac{1}{3}$
Step 3 10 km. $=1$ Mm. (Table II). Then $265.333\frac{1}{3}$ km. $=$ $(265.333\frac{1}{3}\div10)=26.5333\frac{1}{3}$ Mm. Ans.	**Step 3** $265.333\frac{1}{3}\div10=26.5333\frac{1}{3}$

PRACTICE PROBLEMS

1. (a) How many inches in a kilometer? Ans. 39370 in.
   (b) How many centimeters in a mile? Ans. 160934.4 cm.
   (c) How many meters in a furlong? Ans. 201.168 m.

2. The length of a rectangular piece of land is 22.5 rods. Its width is 15.8 rods. How many meters are there in its perimeter? Ans. 385.236+ m.

3. How many pieces, each measuring 6.2 centimeters, can be cut from 223.7 feet of wire? Ans. 1099.7+

Lesson 2

For Step 1, recall the many instances in your everyday life when you have had to measure flat objects; also keep in mind what you have just learned about linear measure. For Step 2, learn the method of computing surfaces or square measures. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Square Measure

A surface has two dimensions—length and width. Take the floor of a room, for instance. It has length and width.

The **area** of a surface is defined as the number of units of surface it contains and is equal to the product of the two dimensions expressed in the same linear unit.

The **unit of surface** is a square, which is a plane figure bounded by four equal sides and having four right angles. (Fig. 2.) A square, each side of which is 1 inch in length, is called a square inch. Squares formed with sides of 1 foot, 1 meter, 1 mile, etc., are called, respectively, 1 square foot, 1 square meter, 1 square mile, etc.

A distinction should be clearly made between the terms "square foot" and "foot square," or between "square mile" and "mile square." In the case of a unit square, such as Fig. 2, there is no difference in the terms used. They are equal to each other, where in this case it is 1 inch on each side and is also 1 square inch or 1 inch square.

On the other hand, Fig. 3, which is 2 inches on each side, is 2 inches square. It may be readily seen, however, that it is not

2 square inches, but 4 square inches in area. Similarly, a ranch, which has a dimension of 3 miles on each side, would have an area of 9 square miles. In other words, we multiply one side by another to get the square units of measure. Be sure that you understand the difference between these two terms.

Square measure, therefore, involves units whose names are the same as those used in the linear tables, with the term "square" in front of each one. The English System has an extra unit used in the measurement of land, which is called the **acre.** This, however, is not a square unit, although it would be approximately

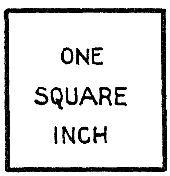


Fig. 2.

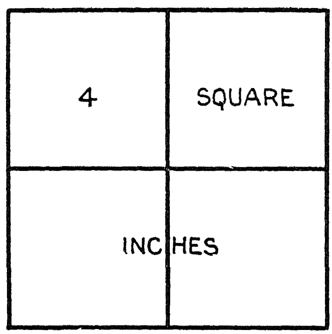


Fig. 3.

equal to the area of a square about 209 feet on each side. Ordinarily, it measures a certain number of rods on one side and a certain number of rods on the other, the total surface being 160 square rods.

In the Metric System, the square centimeter and square meter are used. The larger surfaces are measured as ares and hectares, the former being 10 meters square and the latter equal to 100 ares, or approximately $2\frac{1}{2}$ acres.

TABLE V

Square or Surface Measure

144 square (sq.) in.	= 1 square foot (sq. ft.)
9 square (sq.) ft.	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square (sq.) yds.	= 1 square rod (sq. rd.)
160 square (sq.) rds.	= 1 acre (A)
43560 square (sq.) ft.	= 1 acre
4840 square (sq.) yds.	= 1 acre
640 acres	= 1 sq. mi. = 1 section (sec.)
36 sections	= 1 township (Tp.)

TABLE VI

100 sq. mm.	= 1 sq. cm.	= 1 $cm.^2$
(The ^{2} above is read "square.")		
100 sq. cm.	= 1 sq. dm.	= 1 $dm.^2$
100 sq. dm.	= 1 sq. m.	= 1 $m.^2$ = 1 centare (ca.)
100 sq. m.	= 1 sq. Dm.	= 1 $Dm.^2$
100 sq. Dm.	= 1 sq. Hm.	= 1 $Hm.^2$
100 sq. Hm.	= 1 sq. Km.	= 1 $Km.^2$
100 centares (ca.)	= 1 are (a)	
100 ares	= 1 hectare	

The last two items are land measures and are rarely used in this country.

TABLE VII

Surveyor's Linear Measure		Surveyor's Square Measure	
7.92 in.	= 1 link	16 sq. rd.	= 1 sq. chain
100 links	= 1 chain	10 sq. chains	= 1 acre
80 chains	= 1 mile	640 acres	= 1 sq. mi.

This is called the "Gunter's Chain Table" and is not used so much as formerly. The steel tape is now used and is either 50 or 100 feet in length. This is subdivided into feet and tenths of a foot. This is more convenient as it uses the decimal system.

TABLE VIII

Equivalent Table

Areas

1 sq. in.	= 6.45163 $cm.^2$
1 sq. ft.	= 0.0929 $m.^2$
1 sq. yd.	= 0.83613 $m.^2$
1 $cm.^2$	= 0.155 sq. in.
1 $m.^2$	= 10.76387 sq. ft. = 1.19599 sq. yd.
1 acre	= 40.4687 ares
1 are	= 119.5990 sq. yd.

ILLUSTRATIVE EXAMPLES

1. A ranch is two miles long and three miles wide.
   (a) How many acres in the ranch?
   (b) How many ares in the ranch?

Solution

Instruction	*Operation*
Step 1 Find the number of square miles. There are 6 sq. mi. in the ranch.	**Step 1** $2 \times 3 = 6$
Step 2 1 sq. mi. = 640 acres 6 sq. mi. = $(640 \times 6) = 3840$ acres. Ans.	**Step 2** $6 \times 640 = 3840$
Step 3 1 acre = 40.4687 ares (Table VIII) 3840 acres = $(40.4687 \times 3840) =$ 155399.8+ ares. Ans.	**Step 3** $3840 \times 40.4687 = 155399.8+$

2. A gardener had three rectangular pieces of land. One was 20 rods by 18 rods; another was 7 rods by 33 yards, and the third was 99 feet by 165 feet. (a) How many acres in the three tracts?

Solution

Instruction	*Operation*
Step 1 All measurements must be reduced to same unit. Reduce all to rods (Table I).	**Step 1** 33 yds. = $(33 \div 5\frac{1}{2}) = 6$ rds. 99 ft. = $(99 \div 16\frac{1}{2}) = 6$ rds. 165 ft. = $(165 \div 16\frac{1}{2}) = 10$ rds.
Step 2 Find the number of square rods in each piece and add together. 462 sq. rd. Ans.	**Step 2** $20 \times 18 = 360$ $7 \times 6 = 42$ $6 \times 10 = 60$ 462
Step 3 Find the number of acres. 160 sq. rd. = 1 acre 462 sq. rd. = $(462 \div 160) = 2\frac{71}{80}$ ac. Ans.	**Step 3** $462 \div 160 = 2\frac{142}{160}$ or $2\frac{71}{80}$

A piece of metal is 16 cm. long and 12 cm. wide.
(a) How many square millimeters does it contain?
(b) What is its area in square inches?

Solution

Instruction	*Operation*
Step 1 To find the area in square centimeters, multiply 16 by 12.	**Step 1** $16 \times 12 = 192$
Step 2 Find the area in square millimeters (Table VI). $1\ cm.^2 = 100$ sq. mm. $192\ cm.^2 = (100 \times 192)$ sq. mm. $= 19200$ mm. Ans.	**Step 2** $100 \times 192 = 19200$
Step 3 Find the area in square inches (Table VIII). $1\ cm.^2 = .155$ sq. in. $192\ cm.^2 = (.155 \times 192)$ $= 29.76$ sq. in. Ans.	**Step 3** $.155 \times 192 = 29.760$

PRACTICE PROBLEMS

1. Oscar Mathiesen of Norway set a new world skating record. He made 1500 meters in 2 minutes and 20 seconds. This is a mile in what time? Ans. 2.5+ min.

2. A room is 17 ft. 6 in. by 14 ft. 4 in. What is the square meter area? Ans. 23.3 $m.^2$

3. A barn yard is 100 feet long by 160 feet wide.
   (a) How many square rods in the lot? Ans. 58.7 sq. rd.
   (b) How many square meters in the lot? Ans. 1486.4 $m.^2$

4. A sheet of tin is 15 centimeters square. How many square inches does it contain? Ans. 34.875 sq. in.

Lesson 3

For Step 1, recall what you have learned in Lessons 1 and 2 and also recall the many instances when you have had to measure the amount of material in solid pieces. For Step 2, learn the method of figuring cubic contents. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Cubic Measure

The volume of any solid is obtained by cubic measure. The unit of volume is a cube. (See Fig. 4.) Here you will notice that there are three dimensions—length, width, and thickness. The cube, Fig. 4, may be one inch, or one centimeter, or one foot, on each one of the edges.

The cubical contents of such a figure is called the **volume.** The volume of any such body, where all the angles of the corners are 90 degrees or right angles or square, is equal to the product of its three dimensions expressed in the same linear unit.

In Fig. 4, for instance, we would have a volume of 1 cubic foot if each dimension is one foot. In other words, 1 foot times 1 foot

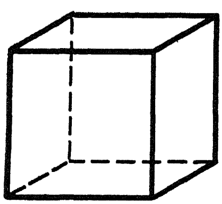


Fig. 4.

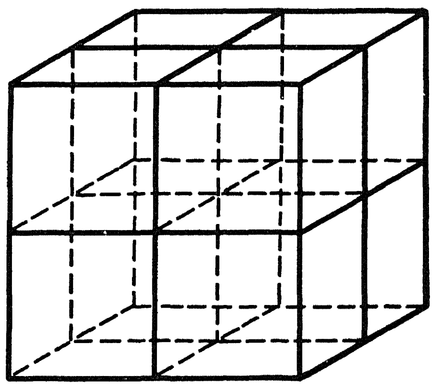


Fig. 5.

times 1 foot equals 1 cubic foot. In case it were 2 feet on each edge it would be 2 times 2 times 2, which would be 8 cubic feet. This is illustrated in Fig. 5.

In case of an irregular body, however, the volume, although still in cubic measure, must be measured by cutting the figure up into small units or by the displacement of liquid, such as water.

When we multiply a number by itself, we say it is raised to the second power; and when we multiply a number by itself and

then the result by that same number, we say it is raised to the third power. Thus, in square measure, when the surface is of the same dimensions on each side, we multiply the side by itself to get the area. In the case of cubic measure in which the three sides are all the same dimension, we raise to the third power the one dimension. Thus, in Fig. 5, we multiply 2 by 2 by 2, which gives us the volume of 8 cubic units.

The units of cubic measure in the English System are the same as those in linear measure with the prefix of the word **cubic.** For example, cubic inch, cubic yard, etc. The units for square measure are the square inch, square yard, etc.

In the Metric System, the cubic centimeter and the cubic meter are commonly used, their relative dimensions being given in the preceding tables. The units for square measure are the square centimeter, square meter, etc.

TABLE IX

Cubic Measure

1728 cubic (cu.) in. = 1 cu. ft.
27 cubic (cu.) ft. = 1 cu. yd.
128 cubic (cu.) ft. = 1 cord of wood
$24\frac{3}{4}$ cubic (cu.) ft. = 1 perch of masonry

1 cu. yd. is called a load of dirt or sand.

TABLE X

1000 cu. mm. = 1 cu. cm. or 1 $cm.^3$ or 1 cc.
1000 cu. cm. = 1 cu. dm. or 1 $dm.^3$ = 1 liter
1000 cu. dm. = 1 cu. meter or 1 $m.^3$ = 1 kilolitre (kl.)

TABLE XI

Equivalent Table

1 cu. in. = 16.387 cc.
1 cu. ft. = 28.317 liters or $dm.^3$
1 cc. = .061 cu. in.
1 liter = 61.0234 cu. in.

In square and cubic dimensions the sign for times (× as used for multiplication) is used a great deal between dimensions as: a box is 10 ft. × 6 ft. × 4 ft. This means that its length is 10 ft.,

its width 6 ft., and its height 4 ft. If these three dimensions are multiplied together, the cubical contents or volume is obtained. A room is 18 ft. × 15 ft. × 8 ft., that is, it is 18 ft. long, 15 ft. wide, 8 ft. high.

ILLUSTRATIVE EXAMPLES

1. A coal bin is 10 ft. × 12 ft. × 14 ft. (a) How many cubic yards does it contain? (b) How many cubic meters does it contain?

Solution

Instruction	*Operation*
Step 1 Find volume in cubic feet by multiplying the three dimensions together. There are 1680 cu. ft.	**Step 1** $10 \times 12 \times 14 = 1680$
Step 2 27 cu. ft. = 1 cu. yd. (Table IX). 1680 cu. ft. = $(1680 \div 27)$ = $62\frac{2}{9}$ cu. yd. Ans.	**Step 2** $1680 \div 27 = 62\frac{6}{27}$ or $62\frac{2}{9}$
Step 3 1 cu. ft. = 28.317 cu. dm. (Table XI). 1680 cu. ft. = (28.317×1680) = 47572.56 cu. dm. 1000 cu. dm. = 1 cubic meter	**Step 3** $28.317 \times 1680 = 47572.56$

47572.56 cu. dm. = $(47572.56 \div 1000) = 47.57256$ cu. m. Ans.

2. A bookcase is 9 in. by 3 ft. 4 in. by 6 ft. 6 in. (a) How many cu. ft. does it contain? (b) How many cu. dm. does it contain?

Solution

Instruction	*Operation*
Step 1 The dimensions must be reduced to the same unit. Reduce them to feet (Table I).	**Step 1** 9 in. $= \frac{9}{12} = .75$ ft. 3 ft. 4 in. $= 3\frac{1}{3}$ ft. 6 ft. 6 in. $= 6.5$ ft.

Step 2	**Step 2**
Multiply the three dimensions together to get volume in cu. ft. 16.25 cu. ft. Ans.	$6.5 \times 3\frac{1}{3} = 21.6\frac{2}{3}$ $21.6\frac{2}{3} \times .75 = 16.25$
Step 3	**Step 3**
1 cu. ft. = 28.317 cu. dm. 16.25 cu. ft. = (28.317 × 16.25) = 460.151 cu. dm. Ans.	$16.25 \times 28.317 = 460.151$

3. A rectangular fish tank is 8 in. by 6 in. by 18 in. (a) How many cubic feet in the tank? (b) How many cubic centimeters?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Multiply the three dimensions to get volume in cu. in. Volume is 864 cu. in.	$8 \times 6 \times 18 = 864$
Step 2	**Step 2**
1728 cu. in. = 1 cu. ft. (Table IX). 864 cu. in. = (864 ÷ 1728) = .5 cu. ft. Ans.	$864 \div 1728 = 0.5$
Step 3	**Step 3**
1 cu. in. = 16.387 cc. (Table XI). 864 cu. in. = (16.387 × 864) = 14158.368 cc. Ans.	$864 \times 16.387 = 14158.368$

PRACTICE PROBLEMS

1. A box is 4 ft. long, 2 ft. wide, and 2 ft. high. What is its volume in cubic inches? Ans. 27,648 cu. in.

2. A laborer digs a ditch 100 ft. long, 18 inches wide, and $2\frac{1}{2}$ ft. deep. How many cubic yards of earth were removed? Ans. $13\frac{8}{9}$ cu. yd.

3. A room is 20 ft. wide, 30 ft. long, and 16 ft. high. How many cubic meters does it contain? Ans. 271.843 $m.^3$

4. If the rainfall on a certain day is $2\frac{1}{4}$ inches, find the number of cubic inches that fall on a piece of ground 25 ft. wide and 100 ft. long. Ans. 810,000 cu. in.

Lesson 4

For Step 1, recall when you have had to measure the amount of liquid in containers, such as buckets, etc. For Step 2, learn the method of figuring measures of capacity. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

MEASURES OF CAPACITY

Capacity signifies the extent of volume or space. In the English System there is no unity in the measurements of capacity because of the use of three kinds of measures.

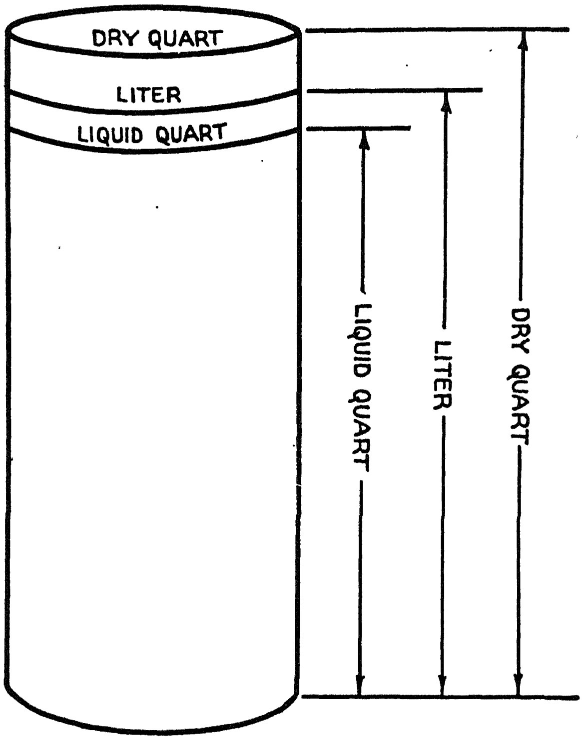


Fig. 6.

There is first the common **liquid measure** used in measuring liquids, as water and milk, and in estimating the capacity of cisterns, reservoirs, flow of water in streams, etc. Then there is the **dry measure** used for measuring grains, vegetables, etc. The third is the **apothecaries' fluid measure** used by drug stores for filling medical prescriptions.

In the Metric System there is only one table for the three uses and therefore it has standard units. The standard unit is the **liter,** which is larger than our liquid quart but smaller than our dry quart.

Fig. 6 shows the comparison of three of the measures.

TABLE XII

Liquid Measure

4 gills (gi.)	=1 pint (pt.)
2 pints (pt.)	=1 quart (qt.)
4 quarts (qt.)	=1 gallon (gal.)
$31\frac{1}{2}$ gallons	=1 barrel (bbl.)
2 barrels	=1 hogshead (hhd.)
1 gallon	=231 cu. in.
$7\frac{1}{2}$ gallons	=1 cu. ft.
1 gal. of water	$=8\frac{1}{3}$ lbs.
1 cu. ft. of water	$=62\frac{1}{2}$ lbs.

TABLE XIII

Dry Measure

2 pints (pt.)	=1 quart (qt.)
8 quarts (qt.)	=1 peck (pk.)
4 pecks (pk.)	=1 bushel (bu.)
1 bu. level full	=2150.42 cu. in.

TABLE XIV

Apothecaries' Measure

60 minims (m.)	=1 fluid dram (f ʒ)
8 fluid drams	=1 fluid ounce (f ℥)
16 fluid ounces	=1 pint (0)
8 pints	=1 gallon (cong.)

TABLE XV

Metric Measure

10 milliliters	=1 centiliter (cl.)
10 centiliters	=1 deciliter (dl.)
10 deciliters	=1 liter (l.) $=$ dm.$^3=1000$ cc.
10 liters	=1 decaliter (Dl.)
10 decaliters	=1 hectoliter (Hl.)
10 hectoliters	=1 kiloliter (Kl.) $=1$ m.^{3}

TABLE XVI

Conversion Table

1 liquid pint	=473.179 cu. centimeters (cc.)
1 liquid quart	=946.358 cu. centimeters (cc.)
1 dry pint	=551.614 cu. centimeters (cc.)
1 dry quart	=1103.228 cu. centimeters (cc.)
1 liter	=61.0234 cu. in. =1000 cc.
1 liter	=2.11336 liquid pints
1 liter	=1.81616 dry pints
1 liter	=1.05668 liquid quarts
1 liter	=.90808 dry quarts

As there is a close association between weights and the above measures, the measures of weight will also be given here.

MEASURES OF WEIGHT

Weight is a measure of the force of the Earth's attraction for a body. In other words, the straight downward pull of the Earth on other objects. This force is called **gravity.**

Weight is measured by **scales** of various kinds. Spring balances are often used for light objects. Other types use lever arms with definite marked weights on one side of the lever.

The English System has three different tables, as in the capacity measures. First are the **avoirdupois weights,** which are used for heavy objects as coal, grain, live stock, meats, groceries, etc. The second are the **Troy weights,** which are used in weighing gold, silver, diamonds, and other precious metals. The third are the **apothecaries' weights,** which are used for weighing drugs and filling prescriptions.

TABLE XVII

Avoirdupois Weight

16 ounces (oz.)	=1 pound (lb.)
100 pounds	=1 hundred weight (cwt.)
20 hundred weight	=1 ton
2000 pounds	=1 ton or short ton
2240 pounds	=1 long ton
196 pounds	=1 barrel of flour
200 pounds	=1 barrel of meat

The **long ton** is used by U. S. Custom houses and in wholesale business in coal and iron.

TABLE XVIII

Troy Weight

24 grains (gr.)	= 1 pennyweight (pwt.)
20 pennyweights	= 1 ounce (oz.)
12 ounces	= 1 pound (lb.)
3.2 grams	= 1 carat

The carat is used in weighing diamonds, etc. The word carat is also used to express the fineness of gold. One carat is $\frac{1}{24}$ part. For instance, gold is called 18 carat when 18 parts are pure gold and 6 parts are other metals, called alloys.

TABLE XIX

Apothecaries' Weight

20 grains (gr.)	= 1 scruple (sc. or ℈)
3 scruples	= 1 dram (dr. or ʒ)
8 drams	= 1 ounce (oz. or ℥)
12 ounces	= 1 pound (lb.)

The pound, ounce, and grain are the same here as in the Troy weights. Wholesale drugs are bought by avoirdupois weights. The avoirdupois pound = 7000 grains and the avoirdupois ounce = 437.5 grains Troy.

25 pounds are sometimes called a **quarter.**

TABLE XX

Metric System

10 milligrams (mg.)	= 1 centigram (cg.)
10 centigrams	= 1 decigram (dg.)
10 decigrams	= 1 gram (g.)
10 grams	= 1 decagram (Dg.)
10 decagrams	= 1 hectogram (Hg.)
10 hectograms	= 1 kilogram (Kg.)
10 kilograms	= 1 myriagram (Mg.)
10 myriagrams	= 1 quintal (q.)
10 quintals	= 1 mittier, tonneau or metric ton (T)

TABLE XXI

Conversion Table

1 grain	= .0648 − grams
1 gram	= 15.432 + grains
1 ounce (avoirdupois)	= 28.3495 grams
1 pound (avoirdupois)	= 453.5924 + grams
1 ton (short)	= 907.185 kilograms
1 kilogram	= 2.2046 (lbs. Av.)
1 metric ton	= 2204.67 (lbs. Av.)

TABLE XXII

Weights per Bushel

Wheat	= 60 lbs.	Clover	= 60 lbs.
Shelled corn	= 56 lbs.	Timothy	= 45 lbs.
Ear corn	= 70 lbs.	Buckwheat	= 52 lbs.
Oats	= 32 lbs.	Apples	= 50 lbs.
Barley	= 48 lbs.	White beans	= 60 lbs.
Rye	= 56 lbs.	Irish potatoes	= 60 lbs.
Blue grass	= 14 lbs.		

There is a difference in the weights of some of the items in the table on grains in different states. The weights given are for a majority of states. The weights in the first column are nearly universal.

ILLUSTRATIVE EXAMPLES

1. A farmer sold his wheat for $1.50 per bushel. He hauled 5 loads of the following net weights: 3000 lbs., 2980 lbs., 3009 lbs., 3025 lbs., and 2986 lbs. How much money did he receive?

Solution

Instruction	*Operatio*
Step 1 First find the total amount of wheat in pounds, by adding the five amounts. This gives 15000 lbs.	**Step 1** 3000 2980 3009 3025 2986 15000

Instruction	*Operation*
Step 2 Find the number of bushels. 60 lbs. of wheat = 1 bushel. 15000 lbs. of wheat = (15000 ÷ 60) = 250 bu.	**Step 2** $15000 \div 60 = 250$
Step 3 Find selling price. 1 bushel sells for \$1.50. 250 bu. sell for (\$1.50 × 250) = \$375. Ans.	**Step 3** $1.50 \times 250 = 375.00$

2. A wholesale grocer bought 2 tons of cheese for \$720 and sold it at 30 cents a pound. What was his gain?

Solution

Instruction	*Operation*
Step 1 Find the number of lbs. in 2 tons (Table XVII). Number of lbs. = 2000 × 2 = 4000 lbs.	**Step 1** $2000 \times 2 = 4000$
Step 2 Find selling price at 30 cents per lb. 1 lb. sells for 30¢. 4000 lbs. sell for (30¢ × 4000) = \$1200.00	**Step 2** $30 \times 4000 = 120000$
Step 3 To find the gain, subtract the cost from the selling price = \$480 Ans.	**Step 3** \$1200 − \$720 = \$480

3. A grocer bought 2 barrels of vinegar at \$20 a barrel. He put this in quart bottles and sold it at 25¢ a bottle. How much money did he make on the 2 barrels?

Solution

Instruction	*Operation*
Step 1 Find the cost price.	**Step 1** Cost = \$20 × 2 = \$40.00

Step 2
Find number of quarts in 2 barrels (Table XII).
2 barrels $= (31\frac{1}{2} \times 2)$ gal.
1 gal. $= 4$ qt.
63 gal. $= (4 \times 63) = 252$ qt.

Step 2

$31\frac{1}{2} \times 2 = 63$
$63 \times 4 = 252$

Step 3
Selling price at 25¢ a quart
$= 252 \times \$.25 = \$63.00$

Step 3

$252 \times .25 = 63.00$

Step 4
Gain is difference between cost and selling price $= \$23$. Ans.

Step 4

$\$63 - \$40 =$

PRACTICE PROBLEMS

1. How many pints are there in $24\frac{1}{2}$ gallons? Ans. 196 pt.
2. At 6 cents a pint, how many gallons can be bought for $3.84? Ans. 8 gal.
3. The water in a reservoir weighs 3750 pounds. How many gallons of water are there in the reservoir? Ans. 450 gal.
4. A grocer buys 5400 pounds of potatoes. He sells them at 27 cents a peck. What was the selling price? Ans. $97.20.

MEASURES OF TIME

The more definite units of time are based on the movements of the heavenly bodies.

The **day** is the length of time required for the Earth to complete one revolution on its axis. The **hour, minute,** and **second** are convenient divisions of the day.

The **week** is the time required for the **Moon** to go one-fourth of the way around the Earth. It therefore has four Quarters as shown in all almanacs.

The **year** is the time required for the Earth to go around the **Sun.**

Due to the Earth's axis varying in respect to the Sun during a revolution, the length of days and nights varies in different times of the year, causing the four seasons—Winter, Spring, Summer, and Autumn. These seasons are the reverse in the northern hemisphere from what they are in the southern hemisphere. For in-

stance, in January we have winter, while in Argentina in January the temperature is reported over 100° or it is midsummer time.

Time is measured the same way all over the world.

As the earth rotates, it is daytime in the United States when it is nighttime in Asia. Also when the sun rises in New York, it

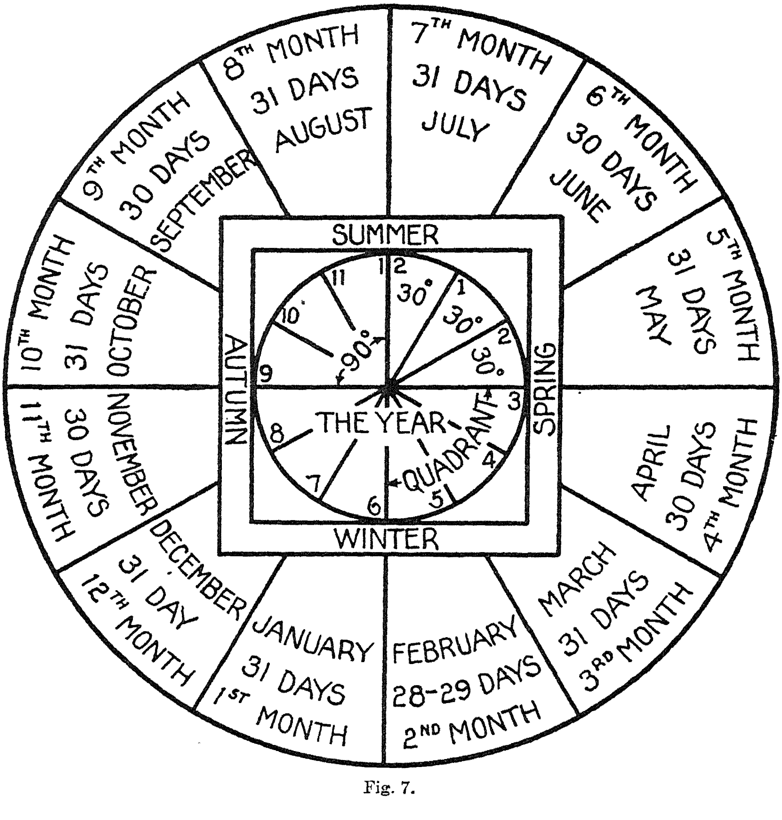


Fig. 7.

is still dark in central and west portions of the United States. This has led to the adoption in the United States of **four standard times,** covering four sections of the country which have one hour's difference in time. They are called **Eastern, Central, Mountain,** and **Pacific.** Thus when it is noon at New York, it is 11 o'clock in Chicago, 10 o'clock in Denver, and 9 o'clock in San Francisco.

In Fig. 7 the divisions of the year are shown graphically.

The lengths of different months have been arbitrarily set and have no significance. At the present time considerable discussion has been started over changing from 12 to 13 months for a year and then equalizing as nearly as possible the number of days for each. The following rhyme will help you to remember the number of days for each month:

Thirty days hath September,
April, June, and November,
All the rest have thirty-one,
Except the second month alone,
To which we twenty-eight assign,
Till leap year makes it twenty-nine.

All years divisible by 4, except century years, are leap years. Also all century years divisible by 400 are leap years. The reason for this is shown by the next to the last item in Table XXIII.

TABLE XXIII

Time Measures

60 seconds (sec.)	=1 minute (min.)
60 minutes	=1 hour (hr.)
24 hours	=1 day (da.)
7 days	=1 week (wk.)
4 weeks	=1 month (mo.)
365 days	=1 year (yr.)
12 months	=1 year
52 weeks	=1 year
30 days	=1 month (banking)
366 days	=1 leap year
1 solar year	=365 da., 5 hrs., 48 min., 49.7 sec.
100 years	=1 century

Another measure closely associated with time measure is **circular measure.** The circle was divided originally into 360 parts, due to the revolution of the Earth around the Sun taking 360 days. The other units have similar names to time units but are measurements of distance. The circular measure table, like the time-table, is used throughout the civilized world. Fig. 7 illustrates circular measure.

TABLE XXIV

Circular Measure

60 seconds (″)	=1 minute (′)
60 minutes	=1 degree (°)
90 degrees	=1 quadrant or right angle
360 degrees	=1 circle or 4 quadrants

MEASURES OF MONEY

Each nation has its own money system. Most students are familiar with the U. S. system.

U. S. System

The U. S. system is based on the decimal system and is easily calculated.

TABLE XXV

10 mills (m)	=1 cent (¢)
10 cents	=1 dime (d)
10 dimes	=1 dollar ($)
10 dollars	=1 eagle (E)
20 dollars	=1 double eagle

There is no coin for a mill. It is used in tax computations chiefly. There are 5¢, 10¢, 25¢, 50¢, and 1-dollar pieces, their values being indicated by their names. These are silver coins except the 5¢ coin, which is made of nickel. In gold, besides the eagle, there is the double eagle ($20.00), the half eagle ($5.00), and the quarter eagle ($2.50). Gold is not much used, especially the smaller coins. Paper money is in common use to take the place of gold. Canadian silver coins have the same face value as U. S. silver coins.

English System

The English unit is the pound sterling and is worth $4.8554 in U. S. money.

TABLE XXVI

4 farthings (far.)	=1 penny (d) =$.0202 U. S. Value
12 pence	=1 shilling (s) =$.2433 U. S. Value
5 shillings	=1 crown =$1.216 U. S. Value
20 shillings	=1 pound or sovereign (£ or Sov.)

Farthings are expressed as fractions of a penny. Thus 1 far. $=\frac{1}{4}$d; 2 far. $=\frac{1}{2}$d; 3 far. $=\frac{3}{4}$d.

The silver coins are the crown = 5s = \$1.212; the half crown = 2s. 6d. = \$0.606; the florin = 2s = \$0.486; the shilling, the sixpence, the fourpence, and the threepence. Copper coins are the penny, half-penny, and farthing.

French System

The money of France is a decimal currency. The unit is the **franc.**

TABLE XXVII

10 millimes	= 1 centime	= \$0.00193
100 centimes	= 1 franc	= \$0.193

The gold coins are the 40, 20, 10, and 5 franc pieces.

The silver coins are the 5, 2, and 1 franc; also the 50 and 20 centime pieces. The bronze coins are the 10, 5, 2, and 1 centime pieces.

German System

The German system is also decimal, the unit being the **mark.**

TABLE XXVIII

100 pfennigs = 1 mark = \$0.2385

The gold coins are the 20, 10, and 5 mark pieces. The silver coins are the 2 and 1 mark and the 20 pfennig pieces. Nickel coins are the 10 and 5 pfennig pieces; and the copper coins are the 2 and 1 pfennig pieces.

MISCELLANEOUS MEASURES

TABLE XXIX

Counting

12 units	= 1 dozen (doz.)
12 dozen	= 1 gross (gro.)
12 gross	= 1 great gross (G. Gro.)
20 units	= 1 score (sc.)

TABLE XXX

Paper

24 sheets	= 1 quire (qre.)
20 quires	= 1 ream (rm.)
2 reams	= 1 bundle (bdl.)
5 bundles	= 1 bale (b)

TABLE XXXI

Books

A sheet folded in 2 leaves is called a folio.
A sheet folded in 4 leaves is called a quarto or 4 to.
A sheet folded in 8 leaves is called an octavo or 8 vo.
A sheet folded in 12 leaves is called a 12 vo.
A sheet folded in 16 leaves is called a 16 vo.
A sheet folded in 18 leaves is called an 18 vo.
A sheet folded in 24 leaves is called a 24 vo.
A sheet folded in 32 leaves is called a 32 vo.

Now you have all the tables in detail and you will find them very handy to refer to after you have finished this text, so keep them for reference in the future.

Lesson 5

For Step 1, keep in mind the tables that have been given and note that it is necessary to change denominate numbers from one denomination to another. For Step 2, learn the method of changing quantities from one denomination to another. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

REDUCTION OF DENOMINATE NUMBERS

It is frequently necessary to change denominate numbers to higher or lower units. This is called **reduction** of the numbers. The following rules indicate the method to use.

Rules. 1. *To change a compound denominate number* (see page 1 for definition), *to a simple number of lower denomination, multiply the quantity of highest denomination in the problem by the number of units of next lower denomination equal to one unit of the quantity of higher denomination in the problem and add this product to the number of lower denomination in consideration. Proceed into lower terms in this manner until the required denomination is reached.*

2. *To change a simple denominate number to a compound number of higher denomination, divide the given number by the number of units contained in one unit of the next higher denomination. If there is any remainder, set it aside and give it the name of the units of*

the dividend. Then in the same manner divide the quotient thus obtained by the number of units contained in one unit of the next higher denomination, and set aside the remainder with its proper name. Proceed in this way until the required denomination is reached. The last quotient and the several remainders, each with its proper name, will be the result sought.

ILLUSTRATIVE EXAMPLES

1. How many feet are there in 6 miles, 116 yards, and 24 feet?

Solution

Instruction	*Operation*
Step 1 Study rule (1) carefully and follow through. First find the number of yards in 6 miles. 1 mile $=1760$ yd. 6 miles $=(1760\times 6)=10560$ yd.	**Step 1** $6\times 1760=10560$
Step 2 Add the 116 yards to the product, giving 10676 yd.	**Step 2** $10560+116=10676$
Step 3 Find the number of feet in 10676 yards. 1 yd. $=3$ ft. 10676 yd. $=(10676\times 3)=32028$ ft.	**Step 3** $10676\times 3=32028$
Step 4 Add the 24 ft. to the product, giving 32052 ft. Ans.	**Step 4** $32028+24=32052$

2. Reduce 765 liquid pints to higher denominations.

Solution

Instruction	*Operation*
Step 1 Read rule (2) and follow through. First find number of quarts in 765 pints. 2 pt. $=1$ qt. 765 pt. $=(765\div 2)=382$ qt. and 1 pint left over as remainder.	**Step 1** $765\div 2=382$ and 1 remainder

Step 2	**Step 2**
Find the number of gallons in 382 quarts.	
4 qt. = 1 gal.	
382 qt. = (382 ÷ 4) = 95 gal. and 2 quarts left over.	382 ÷ 4 = 95 and 2 remainder
95 gal., 2 qt., 1 pt. Ans.	

3. Change 600 acres to square yards.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Find number of square rods in 600 acres.	
1 acre = 160 sq. rd.	
600 acres = (600 × 160) = 96000 sq. rd.	600 × 160 = 96000
Step 2	**Step 2**
Find number of square yards in 9600 square rods.	
1 sq. rd. = $30\frac{1}{4}$ sq. yd.	
96000 sq. rd. = (96000 × $30\frac{1}{4}$) = 2904000 sq. yd. Ans.	96000 × $30\frac{1}{4}$ = 2904000

4. How many cubic inches in 20 cords of wood?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Find number of cubic feet in 20 cords. (Table IX.)	
1 cord = 128 cu. ft.	
20 cords = (128 × 20) = 2560 cu. ft.	20 × 128 = 2560
Step 2	**Step 2**
1 cu. ft. = 1728 cu. in.	
2560 cu. in. = (1728 × 2560) = 4423680 cu. in. Ans.	2560 × 1728 = 4423680

5. Reduce 4900 ounces avoirdupois to higher units.

Solution

Instruction	*Operation*
Step 1	**Step 1**
16 oz. = 1 lb.	
4900 oz. = (4900 ÷ 16) = 306 lbs., and 4 oz. remainder.	4900 ÷ 16 = 306 and 4 remainder
Step 2	**Step 2**
100 lbs. = 1 cwt.	
306 lbs. = (306 ÷ 100) = 3 cwt., and 6 lbs. remainder.	306 ÷ 100 = 3 and 6 remainder
3 cwt., 6 lbs., 4 oz. Ans.	

6. How many eggs in 20 great gross?

Solution

Instruction	*Operation*
Step 1	**Step 1**
1 great gross = 12 gross.	
20 great gross = (12 × 20) = 240 gross.	20 × 12 = 240
Step 2	**Step 2**
1 gross = 12 doz.	
240 gross = (12 × 240) = 2880 doz.	240 × 12 = 2880
Step 3	**Step 3**
1 doz. = 12 units.	
2880 doz. = (2880 × 12) = 34560 units.	2880 × 12 = 34560
34560 eggs. Ans.	

PRACTICE PROBLEMS

1. Reduce 25 tons, 15 cwt., 70 lbs., to pounds. Ans. 51570 lbs.
2. Reduce 2 mi., 192 rods, 2 yds., to feet. Ans. 13734 ft.
3. Express 1507 pints, in gallons, quarts, and pints. Ans. 188 gals., 1 qt., 1 pt.
4. Reduce 285364 seconds to days, hours, minutes, and seconds. Ans. 3 days, 7 hrs., 16 min., 4 sec.

Lesson 6

For Step 1, keep in mind that all these measures that you have studied at times need to be added and subtracted. Also, recall how to add and subtract abstract numbers in order to more easily understand how to add and subtract denominate numbers. For Step 2, learn the method of adding and subtracting denominate numbers. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

ADDITION AND SUBTRACTION OF DENOMINATE NUMBERS

There are two methods of adding and subtracting denominate numbers.

Method I. Reduce the given numbers to the lowest denomination mentioned in the problem, then perform the required operation and reduce the result to higher denominations if necessary.

Method II. Perform the operations on the numbers as they are given, making such reductions as may be necessary during the operations.

Method *II* is ordinarily the more suitable. Note its similarity to addition and subtraction of abstract numbers.

ILLUSTRATIVE EXAMPLES

1. Find the sum of the three lengths: 3 mi., 182 rd., 4 yd., 2 ft.; 304 rd., 1 ft.; 5 mi., 76 rd., 4 yd., 2 ft.

Solution (*Method I*)

Instruction	*Operation*
Step 1	**Step 1**
Reduce the miles to feet.	3 mi. = 5280×3 = 15840 ft.
(1 mi. = 5280 ft.)	5 mi. = 5280×5 = 26400 ft.
Step 2	**Step 2**
Reduce the rods to feet.	182 rd. = 16.5×182 = 3003 ft.
(1 rd. = 16.5 ft.)	304 rd. = 16.5×304 = 5016 ft.
	76 rd. = 16.5×76 = 1254 ft.

Step 3

Reduce the yards to feet.
(1 yd. = 3 ft.)
Set down the given feet.

Step 3

4 yd. =	3×4	=	12 ft.
4 yd. =	3×4	=	12 ft.
			2 ft.
			1 ft.
			2 ft.
			51542 ft.

Step 4

Find total number of feet by adding.

Step 4

Step 5

Reduce 51542 ft. to next higher unit or yards, by dividing by 3, as there are 3 ft. in a yard.
51542 ft. = 17180 yd., 2 ft.

Step 5

$51542 \div 3 = 17180$ and 2 remainder

Step 6

Reduce 17180 yd. to rd. by dividing by $5\frac{1}{2}$, or 5.5, as there are $5\frac{1}{2}$ yd. in a rod.
17180 yd. = 3123 rd., 3.5 yds.

Step 6

$17180 \div 5.5 = 3123$ and 3.5 remainder

Step 7

Reduce 3123 rods to miles by dividing by 320, as there are 320 rods in a mile.
3123 rd. = 9 mi., 243 rd.
Collecting the remainders we get
9 mi., 243 rd., 3.5 yd., 2 ft. Ans.

Step 7

$3123 \div 320 = 9$ and 243 remainder

Solution (*Method II*)

Instruction — *Operation*

Step 1

Set the units of each kind in vertical columns and then add each column separately.

Step 1

mi.	rd.	yd.	ft.
3	182	4	2
0	304	0	1
5	76	4	2
	562		

Step 2

Reduce 5 ft. to yards by dividing by 3.
5 ft. = 1 yd., 2 ft.
Add 1 yd. to 8 yd.

Step 2

$5 \div 3 = 1$ and 2 remainder
8 yd. + 1 yd. = 9 yd.

Step 3

Reduce 9 yd. to rods by dividing by $5\frac{1}{2}$, as
$5\frac{1}{2}$ yd. = 1 rd. 9 yd. = 1 rd., 3.5 yd.
Add 1 rd. to 562 rd.

Step 3

$9 \div 5.5 = 1$ and 3.5 remainder
562 rd. + 1 rd. = 563 rd.

Step 4

Reduce 563 rods to miles by dividing by 320.
563 rd. = 1 mi., 243 rd.
Add 1 mile to 8 miles.
Collecting the different units we have the same results as were obtained in *Method I*.
9 mi., 243 rd., 3.5 yd., 2 ft. Ans.

Step 4

$563 \div 320 = 1$ and 243 remainder
8 mi. + 1 mi. = 9 mi.

2. Subtract 5 bu., 3 pk., 2 qt., 1 pt. from 7 bu., 2 pk., 5 qt.

Solution

Instruction / *Operation*

Step 1

Place units to be subtracted under similar units from which they are to be subtracted.

Step 1

bu.	pk.	qt.	pt.
7	2	5	0
5	3	2	1

Step 2

As 1 pint cannot be subtracted from no pints, take one quart from the quart column (leaving 4 qt.) and change to 2 pt., since 1 qt. = 2 pt.

Steps 2 and 3

bu.	pk.	qt.	pt.
6	6	4	2
5	3	2	1

Step 3

Also, 3 pecks cannot be taken from 2 pecks, so take 1 bu. from the bu. column (leaving 6 bu.). 1 bu. $=4$ pk. Add the 4 pk. to the 2 pk., giving 6 pk.

Step 4

Subtract.

1 bu., 3 pk., 2 qt., 1 pt. Ans.

Step 4

bu.	pk.	qt.	pt.
6	6	4	2
5	3	2	1
1			1

3. An orangeade stand had $4\frac{3}{4}$ gallons of punch from which were sold 50 half pints. How much was left?

Solution

Instruction	*Operation*
Step 1 Reduce $4\frac{3}{4}$ gallons to pints. 2 pt. $=1$ qt. and 4 qt. $=1$ gal., so there are $(2\times4)=8$ pt. in 1 gal.	**Step 1** $4\frac{3}{4}\times8$ or $4.75\times8=38.00$
Step 2 Reduce 50 half pt. to pints, giving 25 pt.	**Step 2** $50\div2=25$
Step 3 Subtract 25 pt. from 38 pt., giving 13 pt.	**Step 3** $38-25=13$
Step 4 Reduce 13 pt. to higher denominations. 13 pt. $=6$ qt., 1 pt. 6 qt. $=1$ gal., 2 qt. 1 gal., 2 qt., 1 pt. Ans.	**Step 4** $13\div2=6$ and 1 remainder $6\div4=1$ and 2 remainder

4. A farmer had a piece of land containing 120 acres, 20 square rods. He sold one corner, measuring 42 rods square. How much land did the farmer have left?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Find area of part sold = 1764 sq. rd.	$42 \times 42 = 1764$
Reduce 1764 sq. rd. to acres.	
1764 sq. rd. = 11 A., 4 sq. rd.	$1764 \div 160 = 11$ and 4 remainder
Step 2	**Step 2**
Place similar units under each other and subtract.	
109 A., 16 sq. rd. Ans.	

acres	sq. rd.
120	20
11	4
109	16

5. Add the following and reduce to highest units: 6 wks., 5 dys., 18 hrs., 30 min.; 49 wks., 20 dys., 5 hrs., 10 min.; 1 yr., 2 wks., 20 hrs.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Place similar units in vertical columns and add.	

yrs.	wks.	dys.	hrs.	min.
	6	5	18	30
	49	20	5	10
1	2		20	
1	57	25	43	40

Step 2	**Step 2**
Reduce 43 hrs. to dys.	
(24 hrs. = 1 dy.)	
43 hrs. = 1 dy., 19 hrs.	$43 \div 24 = 1$ and 19 remainder
Add 1 dy. to 25 dys., giving 26 dys.	
Step 3	**Step 3**
Reduce 26 dys. to wks.	
(7 dys. = 1 wk.)	
26 dys. = 3 wks., 5 dys.	$26 \div 7 = 3$ and 5 remainder
Add 3 wks. to 57 wks., giving 60 wks.	

Step 4 **Step 4**

Reduce 60 wks to yrs.
(52 wks. = 1 yr.)
60 wks. = 1 yr., 8 wks. $60 \div 52 = 1$ and 8 remainder
Add 1 yr. to 1 yr., giving 2 yrs.
2 yrs., 8 wks., 5 dys., 19 hrs., 40 min. Ans.

PRACTICE PROBLEMS

1. Find the sum of 10 yd., 2 ft., 10 in.; 15 yd., 1 ft., 9 in.; 8 yd., 2 ft., 7 in.; 18 yd., 1 ft., 11 in.; 16 yd., 2 ft., 8 in.
Ans. 12 rd., 4 yd., 2 ft., 9 in.
2. Find the sum of 12 A., 35 sq. rd.; 14 A., 110 sq. rd.; 15 A., 132 sq. rd.; 11 A., 96 sq. rd.; 25 A., 100 sq. rd. Ans. 79 A., 153 sq. rd.
3. Find the sum of 5 t., 6 cwt., 14 lbs., 10 oz.; 7 t., 15 cwt., 36 lbs., 15 oz.; 17 t., 5 cwt., 84 lbs., 12 oz.; 70 t., 9 cwt., 94 lbs., 11 oz. Ans. 100 t., 17 cwt., 31 lbs.
4. From 12 gal., 2 qt., 1 pt., 2 gi. take 6 gal., 3 qt., 1 pt., 3 gi.
Ans. 5 gal., 2 qt., 1 pt., 3 gi.
5. From 15 yd., 2 ft., 7 in. take 4 yd., 2 ft., 10 in.
Ans. 10 yds., 2 ft., 9 in.
6. From 25 t., 8 cwt., 75 lbs., 10 oz. take 10 t., 11 cwt., 35 lbs., 15 oz. Ans. 14 t., 17 cwt., 39 lbs., 11 oz.

Lesson 7

For Step 1, keep in mind that measures must be multiplied and divided and that you must know how to multiply and divide abstract numbers in order to multiply and divide denominate numbers. For Step 2, learn the method of multiplying and dividing denominate numbers. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

MULTIPLICATION AND DIVISION OF DENOMINATE NUMBERS

In multiplying compound denominate numbers, the reduction to higher units can be done when multiplying, or afterwards.

ILLUSTRATIVE EXAMPLES

1. Multiply 14 gal., 3 qt., 1 pt. by 7.

Solution

Instruction	*Operation*
Step 1 Set the units down in a line as a multiplicand with the multiplier under them and multiply each unit separately by the multiplier.	**Step 1** 14 gal., 3 qt., 1 pt. 7 98 gal., 21 qt., 7 pt.
Step 2 Reduce 7 pt. to quarts by dividing by 2, since 2 pt. = 1 qt. Add 3 qt. to 21 qt.	**Step 2** 7 pt. ÷ 2 = 3 qt., 1 pt. 21 qt. + 3 qt. = 24 qt.
Step 3 Reduce 24 qt. to gallons by dividing by 4, since 4 qt. = 1 gal. Add 6 gal. to 98 gal. Collecting the results, we have 104 gal., 1 pt. Ans.	**Step 3** 24 qt. ÷ 4 = 6 gal. 98 gal. + 6 gal. = 104 gal.

2. Multiply 6 mi., 109 rd., 10 ft. by 8.

Solution

Instruction	*Operation*
Step 1 Multiply each unit separately by 8.	**Step 1** 6 mi., 109 rd., 10 ft. 8 48 mi., 872 rd., 80 ft.
Step 2 Reduce 80 ft. to rd. by dividing by 16.5 ft., since 16.5 ft. = 1 rd. Add 4 rd. to 872 rd., to get total number of rods.	**Step 2** 80 rd. ÷ 16.5 = 4 rd., 14 ft. 872 + 4 = 876

Step 3	**Step 3**
Divide 876 rd. by 320 to reduce to miles, as 320 rd. = 1 mi.	876 rd. ÷ 320 = 2 mi., 236 rd.
Add 2 miles to 48 miles to get total number of miles.	48 + 2 = 50
Collecting, we have the total result, 50 mi., 236 rd., 14 ft. Ans.	

3. Multiply 9 lb., 14 oz. by 7.

Solution

Instruction	*Operation*
Step 1	**Step 1**
(Use Table XVII.) Set in position for multiplication and multiply each unit by 7 separately.	9 lbs., 14 oz. 7 63 lbs., 98 oz.
Step 2	**Step 2**
Reduce 98 oz. to lbs. by dividing by 16, since 16 oz. = 1 lb.	98 oz. ÷ 16 = 6 lbs., 2 oz.
Add 6 lbs. to 63 lbs.	63 lbs. + 6 lbs. = 69 lbs.
Collecting the units, we have 69 lbs., 2 oz. Ans.	

4. Multiply 55° 40′ 50″ by 20.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Place in position and multiply each unit separately.	55° 40′ 50″ 20 1100° 800′ 1000″
Step 2	**Step 2**
Divide 1000″ by 60 to reduce to minutes.	1000″ ÷ 60 = 16′, 40″
Add 16′ to 800′	800′ + 16′ = 816′

Step 3	**Step 3**
Divide 816′ by 60 to reduce to degrees.	$816' \div 60 = 13°, 36'$
Add 13° to 1100°.	$1100° + 13° = 1113°$
Step 4	**Step 4**
Divide 1113° by 360 to get the number of circles.	$1113° \div 360 = 3$ circles, $33°$
Collecting the results, we have 3 circles, 33°, 36′, 40″. Ans.	

5. Divide 14 gal., 3 qt., 1 pt. by 4.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Divide 14 gal. by 4.	14 gal. $\div 4 = 3$ gal. as quotient and 2 gal. remainder
Step 2	**Step 2**
Reduce the 2 gal. (remainder) to qt.	2 gal. $= (2 \times 4) = 8$ qt.
Add the 8 qt. to the 3 qt. of problem.	$8 + 3 = 11$
Step 3	**Step 3**
Divide the 11 qt. by 4.	11 qt. $\div 4 = 2$ qt. as quotient and 3 qt. remainder
Step 4	**Step 4**
Reduce the 3 qt. to pt.	3 qt. $= (3 \times 2) = 6$ pt.
Add the 6 pt. to 1 pt. of problem.	$6 + 1 = 7$
Step 5	**Step 5**
Divide 7 pt. by 4.	7 pt. $\div 4 = 1$ pt. as quotient and 3 pt. remainder
Step 6	**Step 6**
Reduce 3 pt. to gills.	3 pt. $= (3 \times 4) = 12$ gills
Divide 12 gills by 4.	12 gills $\div 4 = 3$ gills as quotient
Collecting the quotients, we have 3 gal., 2 qt., 1 pt., 3 gills. Ans.	

NOTE. For small numbers it may be easier to reduce to lowest denomination, then perform the division and reduce the quotient to higher denominations, as shown in Example 6.

6. Divide 20 hrs., 12 min., by 6.

$(20\times 60)+12=1212$ min.

1212 min. $\div\ 6=202$ min.

202 min. $\div 60=3$ hrs., 22 min. Ans.

7. A water tank contains 48 cu. ft. A cu. ft. of water weighs 62 lbs., 8 oz. Also 8 lbs., 5.2 oz. equal one gallon. (a) What is the weight of the water in the tank? (b) How many gallons in the tank?

Solution

Instruction	*Operation*
Step 1 1 cu. ft. weighs 62 lb., 8 oz. 48 cu. ft. weigh 48 times 62 lb., 8 oz.	**Step 1** 62 lb. 8 oz 48 2976 lb. 384 oz
Step 2 Reduce 384 oz. to lb. by dividing by 16. 384 oz. = 24 lb.	**Step 2** $384\div 16=24$
Step 3 Add the 24 lb. to the 2976 lb., to obtain total weight. Total weight is 3000 lb. Ans.	**Step 3** $2976+24=3000$
Step 4 8 lb., 5.2 oz. equal 1 gal. The number of gallons in 3000 lb. will equal 3000÷8 lb., 5.2 oz. Before the division can be performed, both quantities must be reduced to the same denomination. Reduce both to ounces. 3000 lb. = $(3000\times 16)=48{,}000$ oz. 8 lb., 5.2 oz. = $(8\times 16)+5.2=133.2$ oz.	**Step 4** 16 8 128 5.2 133.2

Step 5

Divide 48,000 by 133.2
360.36+ gal. Ans.

Step 5

```
133.2)48,000.000(360.36
      39 96
       8 040
       7 992
         480 0
         399 6
          80 40
          79 92
```

8. If the height of the ground floor story of a 10-story building is 15 ft., 8 in. and each of the other stories is 11 ft., 9 in., how high is the building?

There are 9 stories that are 11 ft., 9 in., in height.
Multiply 11 ft., 9 in., by 9 to find height of the 9 stories.
11 ft., 9 in., multiplied by 9 = 99 ft., 81 in. Add 15 ft., 8 in. to this product.
15 ft., 8 in., added to 99 ft., 81 in. gives 114 ft., 89 in.
Reduce 89 in. to feet and inches. 121 ft., 5 in. Ans.

PRACTICE PROBLEMS

Multiply:

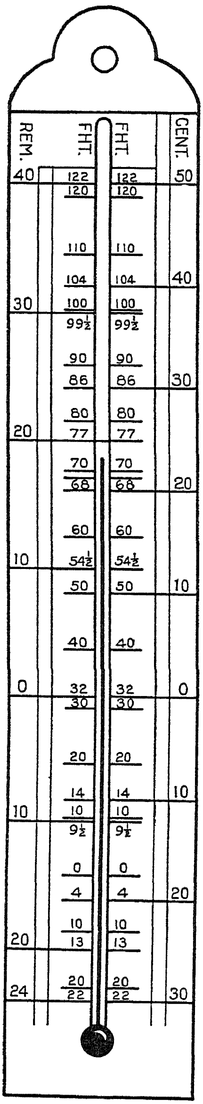
1. 3 hrs., 20 min., 35 sec., by 5 Ans. 16 hrs., 42 min., 55 sec.
2. 2 t., 5 cwt., 48 lb., 15 oz., by 8 Ans. 18 t., 3 cwt., 91 lb., 8 oz.
3. 12 cu. yd., 15 cu. ft., by 6 Ans. 75 cu. yd., 9 cu. ft.

Divide:

4. 15 bu., 3 pk., 5 qt., by 4 Ans. 3 bu., 3 pk., 7 qt., $\frac{1}{2}$ pt.
5. 23 cwt., 68 lb., 10 oz., by 5 Ans. 4 cwt., 73 lb., $11\frac{3}{5}$ oz.
6. 15 rd., 4 yd., 2 ft., 8 in., by 5 Ans. 3 rd., 0 yd., 2 ft., $11\frac{1}{5}$ in.

Lesson 8

For Step 1, bear in mind that heat is a measurable quantity and that there is an instrument to measure heat. For Step 2, learn the method of measuring heat and the method of converting the reading on one kind of thermometer into a corresponding reading on another kind. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.



THERMOMETER SCALES
Fig. 8.

The measurement of heat is very essential in various lines of work.

In domestic science the cook needs a stove which is capable of being regulated accurately.

In foundry work, in welding, in tempering steel, in testing dynamos, and in laboratory experiments, accurate measurements are necessary.

The temperature of living rooms and offices should also be closely measured and regulated.

The instrument used for measuring temperatures is called a **thermometer.**

There are three thermometers in general use. They are named as follows: Fahrenheit (F.), Reaumur (R.), and Centigrade (C.). The centigrade is graded on a decimal scale (scale of 10) and is thus the easiest understood although not used in this country except for laboratory work. Freezing is the zero point (0°) and boiling water is 100 degrees (100°).

The thermometer in common use in the United States is the Fahrenheit. The zero point is 32° below freezing, and the boiling point of water is 212°. Thus the number of degrees from freezing point to boiling point is $212° - 32° = 180°$.

The Reaumur is not seen much in America. The freezing point is zero, the same as the Centigrade, but the boiling point is 80° instead of 100°, so the degrees are larger than in either of the others. Fig. 8 shows a comparison of the three scales. It is well to know how to change the number of degrees on one

type of thermometer to a corresponding number of degrees on another type.

ILLUSTRATIVE EXAMPLES

1. To change from Centigrade to Fahrenheit, multiply the Centigrade degrees by 9, divide by 5, and add 32.

Change 50° Centigrade to degrees Fahrenheit.
$(50\times\frac{9}{5})+32=122°$ F. (See Fig. 8.)

To change from F. to C., reverse the above operation. Subtract 32 from the Fahrenheit degrees, multiply by 5, and divide by 9.

Change 68° F. to degrees C.
$(68°-32°)\times\frac{5}{9}=20°$ C.

2. To change from R. to F., the fraction is $\frac{9}{4}$.

Change 20° R. to F.
$(20°\times\frac{9}{4})+32=77°$ F.

To change from F. to R., just reverse the operation.

Change $99\frac{1}{2}°$ F. to R.
$(99\frac{1}{2}°-32°)\times\frac{4}{9}=30°$ R.

3. To change R. to C., the fraction is $\frac{5}{4}$.

Change 40° R. to C.
$40°\times\frac{5}{4}=50°$ C.

To change from C. to R., just invert the fraction to $\frac{4}{5}$.

Change 50° C. to R.
$50°\times\frac{4}{5}=40°$ R.

NOTE. These fractions are found from the relation between the freezing and boiling points of the thermometers.

$$\text{Thus } 1°\text{ F.}=\frac{100}{180}\text{ C.}=\tfrac{5}{9}\text{ of }1°\text{ C. and }\frac{80}{180}\text{ R.}=\tfrac{4}{9}\text{ of }1°\text{ R.}$$
$$\text{also } 1°\text{ C.}=\frac{180}{100}\text{ F.}=\tfrac{9}{5}\text{ of }1°\text{ F. and }\frac{80}{100}\text{ R.}=\tfrac{4}{5}\text{ of }1°\text{ R.}$$
$$\text{and } 1°\text{ R.}=\frac{180}{80}\text{ F.}=\tfrac{9}{4}\text{ of }1°\text{ F. and }\frac{100}{80}\text{ C.}=\tfrac{5}{4}\text{ of }1°\text{ C.}$$

You see on F. there are $212°-32°=180°$ from freezing point to boiling point; while on C. there are 100° from freezing point to boiling point; and on R. are 80° from freezing point to boiling point. The temperatures below the freezing points are marked with a minus sign, as 20 below zero $= -20°$.

PRACTICE PROBLEMS

1.	Change 50° F. to C.	Ans. 10°
2.	Change 24° R. to F.	Ans. 86°
3.	Change 86° F. to C.	Ans. 30°
4.	Change 30° R. to F.	Ans. $99\frac{1}{2}°$
5.	Change 20° C. to F.	Ans. 68°
6.	Change −30° C. to R.	Ans. −24°

NOTE. Compare results with the Scales, Fig. 8.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on page 47.

If you miss more than two of the problems, it means you should review the whole Section very carefully.

Do not try this trial examination until you have worked every practice problem in this Section.

Do not start the final examination until you have completed this trial examination.

1. If a field is $\frac{1}{4}$ of a mile wide and $\frac{1}{2}$ mile long, how many acres does it contain?

2. A square field has sides each of which is 160 rods long. How many square yards are in this field?

3. One piece of paper is 32 inches square and another piece has an area of 6 square feet. Find the difference in area of the two pieces of paper in square inches.

4. Reduce 2 quadrants, 50 degrees, 40 minutes, and 35 seconds, to seconds.

5. If a printer one day uses 4 bundles, 1 ream, 15 quires, and 20 sheets of paper; the next day, 3 bundles, 1 ream, 10 quires, 10 sheets; and the next, 2 bundles, 13 sheets, how much does he use in the three days?

6. Reduce 15 pounds, 3 crowns, 15 shillings, to U. S. money by table values.

7. A city park board bought the surface soil from a plot of ground 160 rods long and 30 rods wide. They took the surface soil down to a depth of 2 feet. What did the soil cost the board at $0.30 a cubic yard?

8. If the plot of ground described in Problem 7 had to have a fence around it, how much would fencing cost at 40 cents a yard?

9. A box is 4 feet long, 3 feet wide and 2 feet high. How many square feet of surface has it?

10. If one acre of land produces 45 bushels, 3 pecks, 6 quarts and 1 pint of corn, how much will 64 acres produce?

FINAL EXAMINATION

1. A farmer bought a piece of land 220 yards square. How many acres did it contain?

2. What is the difference in area between 20 inches square and 50 square centimeters? Give answer in square millimeters.

3. How many acres are in a square field, each side of which is $\frac{1}{8}$ of a mile long?

4. Reduce 3 miles, 57 rods, 2 yards, 1 foot, 8 inches, to inches.

5. Reduce 157,540 minutes to weeks, days, hours, and minutes.

6. An excavation 58 feet long, 37 feet wide, and 6 feet deep, is to be made for a basement. After 471 cubic feet had been removed how much still remained to be excavated?

7. If a piece of land is 2 kilometers in length and 1.5 kilometers in width, how much would it cost to fence it if the fence cost 25 cents a meter?

8. How many square feet of plastering will be required for the 4 walls and the ceiling of a room 25 feet long, 20 feet wide, and 10 feet high? Make no provision for windows or doors.

9. The temperature according to a centigrade thermometer is 5°. What would this temperature be on a Fahrenheit thermometer?

10. A grocer bought 10 bushels of apples at $1.25 a bushel and sold them at 5 cents a pound. How much profit did he make?

NOTE: Check over all examples carefully to be sure you have made no mistakes.

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. There are several ways in which this problem could be solved. One way is to change the width and length to feet, find the area, and change the area to acres.

One mile equals 5,280 feet. Then $\frac{1}{4}$ mile is $5280 \div 4 = 1{,}320$ feet. Also $\frac{1}{2}$ mile is $5280 \div 2 = 2{,}640$ feet. The area of this field is then $1320 \times 2640 = 3{,}484{,}800$ square feet.

We know that 43,560 square feet equal one acre. Therefore, to find the number of acres in the field, divide 3,484,800 by 43,560.

$$\begin{array}{r} \underline{43560)3484800(80} \\ \underline{348480} \\ 0 \end{array}$$

The last cipher brought down is carried to the answer because 43,560 will not divide into 0. Thus the field contains exactly 80 acres.

2. The area, in rods, of the field is $160 \times 160 = 25{,}600$ square rods. We know that one square rod equals $30\frac{1}{4}$ square yards. Then 25,600 square rods equals $25600 \times 30\frac{1}{4} = 25600 \times 30.25 = 774{,}400$ square yards. Ans.

3. Before we can solve this problem we must understand the difference between "inches square" and "square inches." When we say something is 32 inches square, for example, we mean that it is square in shape and that each of its four sides is 32 inches long. When we say that something has an area of 6 square feet, for example, we do not know what its shape is, nor the length of its sides. All we know is that it has an area of 6 square feet.

To compare the areas of the two pieces of paper in inches we proceed as follows. The first piece is 32 inches square. Thus we know it is square in shape and that each side is 32 inches long. We can find the area by multiplying length by width, or $32 \times 32 = 1{,}024$ square inches.

We already know that the area of the other piece of paper is 6 square feet. In order to compare this with the area of the first piece of paper we must change 6 square feet to square inches. One square foot contains 144 square inches. Then $144 \times 6 = 864$ square inches.

The difference in area between the two pieces of paper, in square inches, is therefore $1024 - 864 = 160$ square inches. In other words, one piece is 160 square inches larger than the other.

4. This problem requires the use of the Table of Circular Measure. We can reduce each item given in the problem to seconds.

The 35 seconds is already in seconds so it stands as is.

The 40 minutes equal $40 \times 60 = 2{,}400$ seconds. This is true because there are 60 seconds in a minute.

The 50 degrees we must first change to minutes and then to seconds. Thus, $50 \times 60 = 3{,}000$ and $3000 \times 60 = 180{,}000$ seconds. This is true because there are 60 minutes in a degree and 60 seconds in a minute.

The 2 quadrants we must first change to degrees, then minutes, and finally seconds. Thus, $2 \times 90 = 180$ and $180 \times 60 = 10{,}800$ and $10800 \times 60 = 648{,}000$ seconds. This is true because one quadrant equals 90 degrees, and 60 minutes equal one degree, etc.

Then $35 + 2{,}400 + 180{,}000 + 648{,}000$ equals 830,435 seconds. Ans.

5. To solve this problem we make use of the Paper Table which is given in the Miscellaneous Measures. We can add-up the various amounts as follows.

4 bundles	1 ream	15 quires	20 sheets	
3 bundles	1 ream	10 quires	10 sheets	
2 bundles			13 sheets	
9 bundles	2 reams	25 quires	43 sheets	(total)

Knowing the total amount of each unit of measure, we can next combine them as follows:

	1	6	
10	$\cancel{3}$	$\cancel{26}$	19
$\cancel{9}$ bundles	$\cancel{2}$ reams	$\cancel{25}$ quires	$\cancel{43}$ sheets

There are 24 sheets in one quire. We can see that 24 goes into 43 once with a remainder of $43-24=19$. Then we add one quire to the 25; make it 26. There are 20 quires in a ream. The 20 will go into 26 once with a remainder of $26-20=6$. We add one ream to the 2, making it 3. There are 2 reams in a bundle. The 2 goes into 3 once with a remainder of 1. We add one to the 9 bundles, making it 10 bundles.

The answer is then 10 bundles, 1 ream, 6 quires, and 19 sheets.

6. Solve the problem using Table XXVI in text.

Step 1. Reduce pounds to dollars.

1 pound $=\$4.8554$

Then 15 pounds equal $\$4.8554\times 15=\$72.8310$

We call this \$72.83.

Step 2. Reduce crowns to dollars.

1 crown $=\$1.216$

Then 3 crowns equal $\$1.216\times 3=\$3.648$

We call this \$3.65.

Step 3. Reduce shillings to dollars.

1 shilling $=\$0.2433$

15 shillings equal $15\times\$0.2433=\$3.6495$

We call this \$3.65.

Add up all steps.

From Step 1..............	\$72.83
From Step 2..............	3.65
From Step 3..............	3.65
	\$80.13

Therefore 15 pounds, 3 crowns, and 15 shillings $=\$80.13$

Note: Step 3 could be solved as follows:

5 shillings $=\$1.216$

Then 15 shillings equal $3\times\$1.216=\$3.648$ because 15 is 3×5. In other words \$1.216 equals 5 shillings, and 5 shillings is a third of 15 shillings. Therefore, 15 shillings $=3$ times $\$1.216=\$3.648$. We call it \$3.65. The result is the same as in Step 3 of the solution.

7. We must find the volume of the soil removed. To find volume we multiply length $\times$ width $\times$ depth. Before this can be done the rods must be changed to feet. We know that one rod equals $16\frac{1}{2}$ feet. The 160 rods equal

$16\frac{1}{2} \times 160 = 2{,}640$ feet. The 30 rods equal $16\frac{1}{2} \times 30 = 495$ feet. Then multiplying $2640 \times 495 \times 2 = 2{,}613{,}600$ cubic feet. This shows how length, width, and depth in feet give volume in cubic feet.

Next the 2,613,600 must be changed to cubic yards. We know that 27 cubic feet equal one cubic yard. Then $2613600 \div 27 = 96{,}800$ cubic yards.

Then 96,800 cubic yards $\times \$0.30 = \$29{,}040.00$. Ans.

8. From the previous solution we know the plot of ground is 2640 feet long and 495 feet wide. In order to find how many yards of fence are required, we must first find the distance around the plot. The plot has four sides. Two sides are 2640 feet long and two sides are 495 feet long. Then $2640 + 2640 + 495 + 495 = 6{,}270$ feet, which is the distance around the plot. There are 3 feet in a yard so $6270 \div 3 = 2{,}090$ yards. Then $2090 \times \$0.40 = \$836.00$. Ans.

9. Such a box has 4 sides, a top, and a bottom, or 6 sides in all.

Two sides are 4 feet by 2 feet, two sides are 3 feet by 2 feet, and two sides are 4 feet by 3 feet. This is easy to understand when we think that the sides which are 4 feet long are 2 feet high and thus measure 4 feet by 2 feet. The sides which are 3 feet long are 2 feet high and thus measure 3 feet by 2 feet. The top and bottom are each 4 feet by 3 feet. If this is not clear, take any small box and mark the length, width, and height as given here and you will be able to visualize it.

The two sides which measure 4 feet by 2 feet are each $4 \times 2 = 8$ square feet. The two sides which measure 3 feet by 2 feet are each $3 \times 2 = 6$ square feet. The two sides which are 4 feet by 3 feet are each $4 \times 3 = 12$ square feet. Adding $8 + 6 + 12$ we have 26 square feet. There are two of each of these sides so $26 \times 2 = 52$ square feet, the total surface. Ans.

If we had such a box with the bottom knocked out, we would figure the surface area in the same way except that we would include only one of the 4 foot by 3 foot sides.

10. To solve this problem, we must use the Dry Measure Table.

First we multiply what one acre will produce by 64.

45 bu.	3 pk.	6 qt.	1 pt.
64	64		64
2,880 bu.	192 pk.	384 qt.	64 pt.

Next we change 64 pints to quarts. There are 2 pints in a quart, so $64 \div 2 = 32$. There is no remainder, so the pints are eliminated and we add the 32 to the 384 quarts, making 416 quarts. There are 8 quarts in a peck, so $416 \div 8 = 52$. There is no remainder, so the quarts are eliminated and we add the 52 to the 192 pecks, making 244 pecks. There are 4 pecks in a bushel, so $244 \div 4 = 61$. There is no remainder, so the pecks are eliminated and we add the 61 to 2,880 making it 2,941 bushels. Ans.

PRACTICAL MATHEMATICS

Section 8

Lesson 1

For Step 1, read carefully the first few paragraphs of the lesson. For Step 2, learn the method of raising numbers to powers. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

POWERS

You learned in the lesson on Factoring that when two or more numbers are multiplied together to give a certain product, these two or more numbers are called factors. Thus $3\times4\times6=72$, or 3, 4, and 6 are factors of 72. $5\times3\times2=30$, or 5, 3, and 2 are factors of 30.

When the factors are all the same number, the product is given a special name, called **power.** Thus $3\times3\times3\times3=81$. The product 81 is the power. The factor 3, in this case, is called the **base,** therefore 3 is the base of the power 81 and 81 is the power of the base 3.

The power, then, is a product obtained by using a base a definite number of times as a factor. In other words, a number, called a base, taken a definite number of times, produces a product, called a power. When the base is used only twice as a factor, the product is called a second power or **square.** Thus, 4×4, or 16, is the square of 4. When the base is used three times as a factor, the product is called the third power or **cube.** $5\times5\times5$, or 125, is the cube of 5.

You learned the use of these words **square** and **cube** in the lesson on Denominate Numbers. You used the word square in finding the area of flat surfaces, while the word cube was used in finding the volume of solids.

When the base is used as a factor more than 3 times, the product is called the fourth power, the fifth power, the sixth power, and so on, according to the number of times the base is used as a factor.

To indicate the number of times the base is to be used as a factor, a small figure is placed above and to the right of the base and is called an **exponent.** For example, 2^4 (read two to the fourth

power), means that the base 2 is to be used 4 times as a factor. Thus, $2\times2\times2\times2$. Now, $2\times2\times2\times2=16$, therefore 16 is the fourth power of the base 2.

This gives a short way of writing several similar factors and indicating their product. The process of finding this product is usually called "raising" the base to the power.

2^2 is read two squared, and means 2×2, which is 4
4^3 is read four cubed, and means $4\times4\times4$, which equals 64
5^4 is read five to fourth power, and means $5\times5\times5\times5$, which is 625

This process of finding the powers of numbers is called **involution.** When powers of fractions are to be found, the numerator and the denominator are handled separately, each being raised to the same power. Thus, $\left(\frac{2}{3}\right)^2$ (read two-thirds squared) $= \frac{2\times2}{3\times3} = \frac{4}{9}$, $\left(\frac{2}{5}\right)^3$ (read two-fifths cubed) $= \frac{2\times2\times2}{5\times5\times5} = \frac{8}{125}$. (Notice that fractions have a parentheses around them besides the power number.) It is seen that the power of a proper fraction is less in value than the base fraction. This of course would not be true of an improper fraction. For example, $\left(\frac{7}{5}\right)^2 = \frac{7\times7}{5\times5} = \frac{49}{25}$, which is greater than $\frac{7}{5}$. (Look up the definitions of proper and improper fractions if you do not remember them.) You can see why this is true from what you learned in the lesson on Fractions.

Decimals or mixed decimals can also have powers. Decimals of course are like proper fractions, the higher the power the less the value. For example, $.2^2$, or $.2\times.2=.04$, which is less than .2; $.3^3$, or $.3\times.3\times.3=.027$, which is less in value than .3.

ILLUSTRATIVE EXAMPLES

1. Find the value of 23^4, or the fourth power of 23.

$$23^4 \text{ means } 23\times23\times23\times23=279{,}841$$

2. Find the fifth power of .03, or $(.03)^5$.

$$(.03)^5=.03\times.03\times.03\times.03\times.03=.0000000243$$

3. Find the fifth power of $\frac{3}{4}$, or $(\frac{3}{4})^5$.

$$(\tfrac{3}{4})^5 \text{ is } \tfrac{3}{4}\times\tfrac{3}{4}\times\tfrac{3}{4}\times\tfrac{3}{4}\times\tfrac{3}{4}=\frac{243}{1024}$$

4. Find the third power of 2.45, or $(2.45)^3$.

This means

$$2.45\times2.45\times2.45=14.706125$$

The number of decimal places in the result may be found by multiplying the number of decimal places in the base by the exponent. In this particular case there are 2×3, or 6 decimal places in the power.

NOTE. In some advanced books on engineering there is found the need for decimal and fractional exponents such as $4^{3.5}$ and $3^{\frac{2}{3}}$. As these cannot be solved by the method of this text, they will be taken up in the lesson on Logarithms.

PRACTICE PROBLEMS

1. Find the value of 2^6. Ans. 64
2. Find the fourth power of 21. Ans. 194,481
3. Raise $\frac{2}{9}$ to the third power. Ans. $\frac{8}{729}$

It will now be necessary to learn how to solve problems like these in the following cases:

ILLUSTRATIVE EXAMPLES

Case (a)—A Power of a Power

1. Raise 3^2 to the third power.

This is expressed in figures by putting a parenthesis around the number in this form $(3^2)^3$. Now, 3^2 is 9, so our problem means that 9 is raised to the third power. Performing this operation gives $9\times9\times9$, or 729. Since 9 is 3 times 3, the problem can be written, $3\times3\times3\times3\times3\times3$. This, you will observe, is 3 to the sixth power, or 3^6. This shows that the exponents can be multiplied together to obtain the required result.

$$(3^2)^3=3^{2\times3}=3^6$$

In the same way $(2^3)^4$ can be written $2^{3\times4}=2^{12}$, or 2 to the twelfth power.

2. Find the value of $(10^3)^4$.
$(10^3)^4=(10\times10\times10)^4$, or $(1000)^4$, which is $1000\times1000\times1000\times1000=1{,}000{,}000{,}000{,}000$. This is 1 with 12 ciphers, or 10^{12}, which $=10^{3\times4}$, or 10 raised to a power equal to the product of the two exponents. Thus when a base with an exponent is raised to a power, the two exponents are multiplied together for a new exponent which indicates the number of times the base is used as a factor.

Case (b)—Multiplication of Powers

3. Multiply 10^3 by 10^4.
Since $10^3=1000$, and $10^4=10000$, our problem is the same as 1000 multiplied by 10,000, which is 10,000,000. That is, a number consisting of 1 and 3 ciphers multiplied by a number composed of 1 and 4 ciphers gives as a product a number consisting of 1 and 7 ciphers, or 10^7. From this it is seen that we added the two original exponents.

$$10^3\times10^4=10^{3+4}, \text{ or } 10^7$$

Similarly $4^2\times4^3\times4^5=4^{2+3+5}$, or 4^{10}, and $(21.5)^3\times(21.5)^5=(21.5)^{3+5}$, or $(21.5)^8$.

Case (c)—Division of Powers

4. Divide 10^7 by 10^4.
Here the reverse of multiplication is necessary. You have 1 with 7 ciphers divided by 1 with 4 ciphers. By cancellation $\frac{10^7}{10^4}=\frac{10000000}{10000}=1000=10^3$. Here the exponents have been subtracted. In other words

$$10^7\div10^4=10^{7-4}=10^3$$

Similarly $(450)^4\div(450)^2=(450)^{4-2}=(450)^2$; $(64)^5\div(64)^3=(64)^{5-3}=(64)^2$

PRACTICE PROBLEMS

1. Find the value of $(2^4)^3$ — Ans. 4096 ✓
2. Find the value of $(3^3)^2$ — Ans. 729
3. Find the value of $3^3\times3^4$ — Ans. 2187
4. Find the value of $(.02)^3\times(.02)^3$ — Ans. .000000000064
5. Find the value of $(2.2)^5\div(2.2)^2$ — Ans. 10.648
6. Find the value of $750^8\div750^6$ — Ans. 562500
7. Find the value of $\left(\frac{2}{5}\right)^2\times\left(\frac{2}{5}\right)^3$ — Ans. $\frac{32}{3125}$

Lesson 2

For Step 1, recall the previous lesson on powers, and division of numbers. For Step 2, learn the method of extracting roots. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

ROOTS

A **root** of a number is one of the **equal factors,** which, when multiplied together, give the number. Therefore, one of the equal factors thus used to obtain a power is a root of the power. For example, the two equal factors of 16 are 4 and 4 ($16=4\times4$). So 4 is a root of 16. Similarly the two equal factors of 81 are 9 and 9 ($81=9\times9$). So 9 is a root of 81. Since there are two equal factors or roots, either one is called the **square root.**

The process of finding one of the two equal factors of a number is called **extracting the square root.**

In case the number has three equal roots or factors, the process is called finding the **cube root.** For example, $27=3\times3\times3$, so 3 is the cube root of 27.

The general process of finding roots of numbers is called **evolution.**

The symbol used to indicate the root of a number is $\sqrt{}$ and is called the **radical sign.** The number of which the root is to be found is placed under the sign.

When cube root is to be indicated, a small figure 3 is placed above the radical sign, thus $\sqrt[3]{}$. $\sqrt[3]{27}$ therefore means the cube root of 27. When the fourth root is indicated a small figure 4 is similarly placed. Thus $\sqrt[4]{81}$ means the fourth root of 81. If higher roots are to be indicated, other figures are used accordingly. The fifth root is indicated thus, $\sqrt[5]{}$, the sixth root, $\sqrt[6]{}$, etc.

This small figure is called the **index** of the root. When the square root is indicated, the small figure is omitted, so the radical sign when used without an index is understood to mean square root. Thus $\sqrt{16}$ means the square root of 16; $\sqrt{144}$ means the square root of 144.

In this text only the method for finding the square root will be discussed, due to the difficulties of finding the higher roots. In the lesson on Logarithms, however, an easy method will be shown for finding all roots. This method will also easily solve large powers.

Where square roots of numbers which have only one, two, and sometimes three figures are to be found, you can easily pick out the root without any special process. In the case of large numbers, however, it will be necessary to have a special process, a description of which follows.

Process of Finding Square Root of Whole Numbers

ILLUSTRATIVE EXAMPLES

NOTE. Be sure to actually do each step as you read it. Do not merely follow through the solution as it is given here, but put down yourself each figure and step of the process as it occurs in the solution. Only in this way will you be able to learn the method of finding the square root. No attention need be given to the principles underlying the process nor to the reasons for using the various steps. Just take these for granted and memorize the process itself.

1. Find the square root of 1369.

Solution

Instruction	*Operation*
Step 1 Separate the number into periods of two figures each, beginning at the right, and place a curved line over each as shown. The number of periods thus formed will be the same as the number of figures in the answer.	**Step 1** $\overset{\frown}{13}\ \overset{\frown}{69}$
Step 2 Next draw a line straight up and down at the left of the number, and a broken line to the right of number, as shown in the illustration.	**Step 2** $\overset{\frown}{13}\ \overset{\frown}{69}$
Step 3 Look at the first period at the left end with the little curve over it and see if you can decide what number multiplied by itself (or	**Step 3** $\overset{\frown}{13}\ \overset{\frown}{69}\,\underline{\vert\, 3}$ 9 469

squared) will equal it, or a little less, but not more. In this case the period is 13. You can see that 4×4 equals 16, that is too much. The next smaller number is 3, and 3×3 equals 9. This number is smaller than 13, so 3 is the number to be used as the first figure of the root and you put it at the right of the number above the curved line as the first figure of the root. Square this number and place the result under the first period and subtract. The square of 3 is 9 and you place 9 under the 3 of the 13. Now subtract 9 from 13 and bring down the next period $\widehat{69}$, and place it beside the remainder 4 as shown. This gives 469.

Step 4

Take 3, the first figure of the root, multiply it by 2, and put the product 6 to the left of 469 back of the up and down line. The 6 is called the **trial divisor.**

Step 4

```
    13 69 |3
     9
6  | 469
```

Step 5

Find now how many times 6 is contained in 46, the first two figures of 469. It is contained 7 times. This 7, then, is the second figure in the root. Place this figure in the root and also at the right of the 6 (or trial divisor), giving 67, which is called the **complete divisor,** back of the up and down line. Now multiply the 67 by the 7 just put in the root and write the product 469 under the number 469. Since there is no remainder, the square root of 1369 is 37. To check your work, square your root. 37×37 =1369. **Step 5** shows the complete operation.

Step 5

```
    13 69 |37 root
     9
67   469
     469
```

2. Find the square root of 273529.

Solution

Instruction | *Operation*

Step 1

Separate the number into periods of two figures each, beginning at the right, and place a curved line over each period as shown.

Step 1

$$\overset{\frown}{27}\ \overset{\frown}{35}\ \overset{\frown}{29}$$

Step 2

Draw a straight line up and down at the left of the number and a broken line to the right (to contain the root) as shown.

Step 2

$$\overset{\frown}{27}\ \overset{\frown}{35}\ \overset{\frown}{29}\,|$$

Step 3

Look at the first period at the left, which is 27, and decide what number multiplied by itself (or squared) will equal it, or be a little less but not more. Try 5. $5\times5=25$, which is less than 27, so put 5 as the first figure in the root. Square the 5 and put the result, 25, under the 27, and subtract. This leaves a remainder 2. Put down the next period, 35, to the right of the 2, giving 235.

Step 3

$$\begin{array}{|l|l} \overset{\frown}{27}\ \overset{\frown}{35}\ \overset{\frown}{29} & \underline{5} \\ 25 & \\ \quad\overline{2}\ 35 & \end{array}$$

Step 4

Multiply 5, the first figure of the root, by 2 and put the product, 10, to the left of 235 back of the up and down line, as the trial divisor.

Step 4

$$\begin{array}{r|l|l} & \overset{\frown}{27}\ \overset{\frown}{35}\ \overset{\frown}{29} & \underline{5} \\ & 25 & \\ 10 & \quad\overline{2}\ 35 & \end{array}$$

Step 5

Find how many times 10 is contained in 23 (the first two figures of 235). It is contained 2 times. The 2 will be the second figure in the root. Place the 2 beside the 5 in the root and also at the right of the 10 (or trial divisor), giving 102 as complete divisor. Now multiply 102 by the 2 just put in the root and place the product 204 under the 235, and subtract. The remainder is 31. Bring down the next period, 29, and place it to the right of the 31, giving 3129.

Step 5

```
        27 35 29 | 52
        25
         2 35
102      2 04
           31 29
```

Step 6

Multiply the 52, which is now in the root, by 2, and put the product, 104, to the left of 3129 back of the up and down line as the new trial divisor.

Step 6

```
        27 35 29 | 52
        25
102      2 35
         2 04
104        31 29
```

Step 7

Find how many times 104 is contained in 312 (the first three figures of 3129). It is contained 3 times. So put 3 as the third figure of the root and also place it at the right of the 104, giving 1043 as the complete divisor. Multiply 1043 by the 3 just put in the root and put the product, 3129, under the 3129. There is no remainder. The required square root is 523.

Step 7

```
         27 35 29 | 523
         25
          2 35
102       2 04
            31 29
1043        31 29
```

3. Find

Solution

Instruction	*Operation*
Step 1 Separate the number into periods of two figures each, beginning at the right, and place a curved line over each period. The left-hand period has only one figure.	**Step 1** $\overset{\frown}{5}\ \overset{\frown}{31}\ \overset{\frown}{76}\ \overset{\frown}{36}$
Step 2 Draw a vertical line at the left of the number and a broken line in which to place the root.	**Step 2** $\vert\ \overset{\frown}{5}\ \overset{\frown}{31}\ \overset{\frown}{76}\ \overset{\frown}{36}\ \underline{\vert\quad}$
Step 3 Consider the first period to the left, which is 5, and decide what number multiplied by itself (or squared) will give 5 or a little less. The number is 2. Put 2 as the first figure in the root. Square the 2 and place the result, 4, under the 5, and subtract. The remainder is 1. Place the next period, 31, to the right of 1.	**Step 3** $\overset{\frown}{5}\ \overset{\frown}{31}\ \overset{\frown}{76}\ \overset{\frown}{36}\ \underline{\vert\ 2\ }$ 4 1 31
Step 4 Multiply 2, the figure in the root, by 2 and put the product, 4, as trial divisor, back of the vertical line.	**Step 4** $\overset{\frown}{5}\ \overset{\frown}{31}\ \overset{\frown}{76}\ \overset{\frown}{36}\ \underline{\vert\ 2\ }$ 4 1 31
Step 5 Find how many times 4 is contained in 13. It is contained 3 times. Put this 3 beside the 2 already in the root and also at the right of the trial divisor, 4. This gives 43 as the complete divisor.	**Step 5** $\overset{\frown}{5}\ \overset{\frown}{31}\ \overset{\frown}{76}\ \overset{\frown}{36}\ \underline{\vert\ 23\ }$ 4 43 1 31 1 29 2 76

Multiply this complete divisor by the 3 just put in the root and place the product, 129, under 131, and subtract. Remainder is 2. Place the next period, 76, beside the remainder.

Step 6

Multiply the 23 in the root by 2, and put the product, 46, as trial divisor, back of the vertical line.

Step 6

```
      5 31 76 36| 23
      4
  43  1 31
      1 29
  46    2 76
```

Step 7

How many times is 46 contained in 27, the first two figures of remainder? It is contained 0 times. So 0 is the third figure to put in the root. Put the 0 also beside the trial divisor, giving 460 as the complete divisor. Multiply this complete divisor by 0, putting the product under the 276, and subtract. Remainder is 276. Put the next period 36 beside the 276.

Step 7

```
      5 31 76 36| 230
      4
      1 31
  43  1 29
        2 76
 460    0 00
        2 76 36
```

Step 8

Multiply the 230 of the root by 2 and put the 460 as the trial divisor back of the vertical line.

Step 8

```
      5 31 76 36| 230
      4
      1 31
 .43  1 29
        2 76
 460    0 00
        2 76 36
```

Step 9
How many times is the trial divisor 460 contained in 2763 (the first four figures of the remainder)? It is contained 6 times. 6 is then the fourth figure in the root. Place the 6 also beside the trial divisor, giving 4606 as complete divisor. Multiply 4606 by the 6 just put in the root and place the product under the 27636, and subtract. 2306 is the required root.

Step 9

```
          5 31 76 36 | 2306
          4
   43   | 1 31
        | 1 29
        |   2 76
  460   |   0 00
        |   2 76 36
 4606   |   2 76 36
```

4. Find the square root of 570.7321

Solution

Instruction | *Operation*

Step 1
When the root of a number containing a decimal is to be found, the division of the number into periods is done by starting at the decimal point and marking off in both directions from the decimal point. (The two figures of a period must never be separated by a decimal point.)

Step 1

```
5 70.73 21
```

Step 2
Draw the vertical line and the broken line to contain the root. Proceed with process of finding root, disregarding decimals.

Step 2

```
| 5 70.73 21 |___
```

Step 3
The first period is 5, and it is readily seen that 2 will be the first figure of the root. Square the 2 and subtract the result, 4, from 5, leaving 1. Bring down the next period 70 beside the 1.

Step 3

```
| 5 70.73 21  2
| 4
| 1 70
```

Step 4
Multiply the 2 in the root by 2, and place the product, 4, as the trial divisor.

Step 4

```
      5 70.73 21 | 2
      4
  4   1 70
```

Step 5
How many times is the trial divisor, 4, contained in 17, the first two figures of the remainder? It is contained 4 times. The second figure of the root would then seem to be 4, and the complete divisor 44. If we multiply 44 by 4, we get 176, which cannot be subtracted from 170 so we must use a smaller number than 4 in the root. Try 3. Put 3 as the second figure of the root, and also beside the 4, giving 43 as the complete divisor. Multiply 43 by the 3 just put in the root, and place the product 129 under 170, and subtract. The remainder is 41. Bring down the next period 73 beside the 41.

Step 5

```
      5 70.73 21 | 23

 43   1 70
      1 29
        41 73
```

Step 6
Multiply the 23 in the root by 2, giving 46 as trial divisor.

Step 6

```
      5 70.73 21 | 23
      4
 43   1 70
      1 29
 46     41 73
```

Step 7
How many times is 46, the trial divisor, contained in 417? It is contained 9 times. But if we used 9 as the third figure of the root, we should find (as we did in **Step 5**) that it would be too large. So try 8, and proceed as in **Steps 5, 6 and 7,** to find the remaining figures of the root.

Step 7

```
       | 5 70.73 21   23.89
       | 4
   43  | 1 70
       | 1 29
       |   41 73
  468  |   37 44
       |     4 29 21
 4769  |     4 29 21
```

We must now place the decimal point in the root. To do this it is necessary only to remember that there will be as many figures

in the whole number in the root as there are periods in the whole number of the number of which we are finding the root; and there will be as many figures in the decimal part of the root as there are periods in the decimal part of the original number. There are two periods in the whole number 5 70; so there will be 2 figures in the whole number of the decimal. There are two periods in the decimal part, 73 21, so there must be two figures in the decimal part of the root. The required root is, then, 23.89.

5. Find the square root of 2.916.

Solution

Instruction

Here we have an odd number of figures in the decimal part. In such a case we must annex a cipher to the decimal part so that there will be an even number to separate into periods.

Operation

	2.91 60	1.70
1	1	
27	1 91	
	1 89	
340	2 60	
	0 00	
	2 60 remainder	

There are two periods in the decimal part of the original number so there must be two figures in the decimal part of the root, thus 1.70+, If we wished to carry the decimal part farther than two places. we should just annex periods of two ciphers each. (2.91 60 00 00)

6. Find the square root of .001225.

Solution

Instruction

It will be seen that the first period contains only zeros, so 0 will be the first figure in the root. Begin by finding the number which when squared will give a little less than 12. It is 3. So place 3 beside the 0 already in the root and proceed as in previous examples. Since there are three periods in the decimal part of the original number, there must be three figures in the decimal part of the root. The required root then is .035.

Operation

	.00 12 25	.035
	9	
65	3 25	
	3 25	

Square Root of Fractions

If both the numerator and denominator are themselves the squares of numbers, just find the square root of the numerator and the denominator separately. For example, find the square root of $\frac{625}{2500}$. The square root of 625 is 25; the square root of 2500 is 50; so

$$\sqrt{\frac{625}{2500}}=\frac{25}{50}$$

If both the numerator and denominator are not perfect squares, reduce the fraction to lowest terms (see Fractions Part 1), then change to a decimal by dividing the numerator by the denominator finally find the square root of the decimal number.

PRACTICE PROBLEMS

1. Find the square root of 1444	Ans. 38
2. Find the square root of 855625	Ans. 925
3. Find the square root of 1.8769	Ans. 1.37
4. Find the square root of 7.365	Ans. 2.71+
5. Find the square root of .0563	Ans. .23+
6. Find the square root of $\frac{2025}{3000}$	Ans. .82+

You will sometimes find fractional exponents as $10^{\frac{3}{2}}$. This means that the square root is to be found for the cube of 10, or $\sqrt{10^3}$, or $\sqrt{1000}$.

The numerator of the fractional exponent indicates the power while the denominator of the exponent indicates the root.

The exponent may be either a proper or an improper fraction and of any size. In case the figures are large, as $85^{\frac{7}{8}}$, then the only way to solve them is by methods described in the lesson on Logarithms. Such problems, however, are not as common as square root, which is often used in everyday life.

If you have the area of a **square surface,** you can find the length of one side by extracting the square root of the area, for one side squared equals the area since the sides of a square are equal.

REVIEW

At this time review Pages 7 and 12 of Section 7. There it explains the relations of **Linear Measurements** to **Square Areas** and **Cubic Volumes.** Let us review this information.

(A) A **Plane Surface** has two dimensions—width and length. The product of these two dimensions equals the area in **Square Units.** It makes no difference whether the surface is square, rectangular, or circular. In case of the square, the area equals the product of length by width or one side squared, as the width equals the length. In case of a rectangle, the area is the product of length by width. In case of a circle, the area is the product of a constant π, called **Pi,** and the radius squared. This subject is taken up in a later Section.

(B) A **Solid Surface** has three dimensions—width, length, and thickness. The product of these three dimensions equals the volume in **Cubic Units.**

Let us use as a concrete example a schoolroom which is 40 feet wide, 40 feet long and 10 feet high.

The floor is 40 feet $\times$ 40 feet = 1600 Square Feet Area

Each side wall is 40 feet $\times$ 10 feet = 400 Square Feet Area

Each end wall is 40 feet $\times$ 10 feet = 400 Square Feet Area

The volume of the room is the contents or amount of air contained.

40 feet $\times$ 40 feet $\times$ 10 feet = 16000 Cubic Feet

Now if you are given the area of the floor and the length, you can find the width by division, for 1600 square feet divided by 40 feet equals 40 feet. Also since these two dimensions are equal, you can find the dimensions with only the area given, for the square root of 1600 square feet equals 40 feet, which is the width as well as the length. *If you have the volume and height given, then you can find the product of the width and length for 16,000 cubic feet divided by 10 feet equals 1600 square feet. From this, as shown above, you can find both length and width.* Therefore, if the container or solid has two equal dimensions and you have the volume and the third or unequal dimension given, then you can solve the problem for the two equal dimensions.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on page 19.

If you miss more than two of the problems it means you should review the whole Section very carefully.

Do not try this trial examination until you have worked every practice problem in the Section.

Do not start the final examination until you have completed this trial examination.

1. A square cellar had 1200 cubic yards of dirt removed to make it 3 yards deep. What is the length *in feet* of one side of the cellar?

2. Extract the square root of .369 to 4 decimal places.

3. Find the square root of:

(a) $\frac{1}{8} \div \frac{1}{5}$ Give answer in two decimal places.
(b) $\frac{30}{32} \div 5$ Give answer in three decimal places.

4. Find the square root of:

(a) 112225 (b) .012996

5. A kitchen has an area of 196 square feet. If the kitchen is square in shape, how many feet of border for the outside edge will the linoleum man have to have?

6. Find the square root of:

(a) 20.7936 (b) $17\frac{3}{8}$ Have four decimal places in answer.

7. A piece of linoleum is 4 times as long as it is wide. The area of the whole piece is 400 square feet. If this linoleum can be marked off into 4 square pieces so that all four pieces are exactly the same size, what are the dimensions of the whole piece?

8. Find the square root of:

$(\frac{2}{5})^2 \times \frac{5}{4}$ Answer must have three decimal places.

FINAL EXAMINATION

1. Find the value of:

(a) $(3^3 \times 4^2)^2$ (b) $(\frac{2}{3} \text{ of } \frac{6}{7})^3$

2. Find the value of each of the following:

(a) $(2.61)^2$ (b) $(.032)^3$

3. Find the square root of:

(a) 2401 (b) 1179.9225

4. Find the square root of each, to the fourth decimal place.

(a) $\frac{123}{64}$ (b) .00098596

5. A square park containing $14\frac{2}{5}$ acres is to be fenced. How many yards of fence will be required?

6. The coal in a square coal bin was 10 feet deep. There were then just 885.0625 cubic feet of coal in the bin. What was the length of one side of the bin? Give answer correct to second decimal.

7. A tunnel with a square section was excavated through 150 feet of clay, and 21,600 cubic feet of clay were removed. What was the width of the tunnel?

8. A man has 200 yards of carpeting $1\frac{1}{8}$ yards wide. If this carpet just covers the floor of a square room, what is the length of one side of the room in feet?

Note: Check over your work again to make sure you have not made mistakes. Review the lessons to make sure you used the principles correctly.

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. We can easily reason out how to solve this problem if we remember what must be done in calculating volume. If we know the length, width, and depth of anything, we find its volume by multiplying length×width×depth. In this problem we are given the volume and we know the depth. When we multiply length by width we obtain area. Then we multiply the area by the depth to find volume. Therefore, if we divide the volume by the depth we can obtain the area.

In this problem the volume is 1200 cubic yards. Dividing 1200 by the depth (3 yards) we have

$$1200 \div 3 = 400$$

This 400 is then the area of the cellar.

We know the cellar is square in shape, so all four sides are equal. Also we know that area equals length×width. Or, to get the area of the cellar, its length had to be multiplied by its width. Seeing that all sides are equal, the length and width will be the same. Therefore, if we find the square root of 400 we will have the length of one side.

This is 20 yards because the original volume and depth were given in yards. Then

$$20 \times 3 = \textit{60 feet.} \quad \text{Ans.}$$

2. The number of which we must find the square root is a pure decimal. The marking off of periods is done starting at the decimal point and marking off in both directions. There is no whole number in this problem, so we mark off periods to the right of the decimal point, making a period for each decimal place desired in the answer. This problem calls for 4 places in the answer, so we add 5 ciphers to make up the required number (4) of periods.

	.36 90 00 00	/.6074
	36	
1207	90 00	
	84 49	
12144	5 51 00	
	4 85 76	

The first period is 36 and as 6 is the largest number whose square will divide into this period the 6 becomes the first figure in the answer. Square 6 and subtract from 36. Bring down the next period, which is 90. The 6 multiplied by 2 becomes the trial divisor. It can be seen that the trial divisor cannot be divided even once into 90 because the complete divisor would be 121. Therefore we add a cipher in the answer and bring down another period, making 9000. We again multiply the answer by 2 and have 120. After a few trials we find that 1207 will divide into 9000 so the 7 is put in the answer. Then $7 \times 1207 = 8449$. After subtracting we have 551. Another period is brought down and the process repeated.

The root (answer) is .6074. In pointing off the decimal point in the answer we count the decimal places in the number we found the square root of. There are eight places. In the answer we point off only half this many. This can be remembered as a rule. We start at the right hand end of the answer and count four places. The answer is then .6074. Another way to point off is to have as many decimal places in the answer as there are periods to the right of the decimal in the number which you are finding the square root of.

3. (a) To find the square root of $\frac{1}{8} \div \frac{1}{5}$ we must first do the dividing.

$$\tfrac{1}{8} \div \tfrac{1}{5} = \tfrac{1}{8} \times \tfrac{5}{1} = \tfrac{5}{8}$$

Next we change $\frac{5}{8}$ to a decimal.

$$\tfrac{5}{8} = 5 \div 8 = .625$$

Next find the square root of .625

$$\begin{array}{rl}
 & |\overset{\frown}{.62}\ \overset{\frown}{50}\ \ /\underline{.79}\ \ \text{(Ans.)} \\
 & \ 49 \\
149 & \ 13\ 50 \\
 & \ 13\ 41
\end{array}$$

(b) Here we must first divide $\frac{30}{32}$ by 5.

$$\frac{30}{32} \div 5 = \frac{30}{32} \div \frac{5}{1} = \underset{16}{\frac{\cancel{30}}{\cancel{32}}} \times \underset{1}{\frac{1}{\cancel{5}}} = \frac{3}{16}$$

Next change $\frac{3}{16}$ to a decimal.

$$\tfrac{3}{16} = 3 \div 16 = .1875$$

Next find the square root of .1875

$$\begin{array}{rl}
 & |\overset{\frown}{.18}\ \overset{\frown}{75}\ \overset{\frown}{00}\ \ /\underline{.433}\ \ \text{(Ans.)} \\
 & \ 16 \\
83 & \ \ 2\ 75 \\
 & \ \ 2\ 49 \\
863 & \ \ \ \ 26\ 00 \\
 & \ \ \ \ \underline{25\ 89} \\
 & \ \ \ \ \ \ \ 11
\end{array}$$

Usually, in cases of this kind, three decimal places in the answer are enough.

4. (a)

$$\begin{array}{rl}
 & |11\ 22\ 25\ \ /\underline{335}\ \ \text{(Ans.)} \\
63 & \ 2\ 22 \\
 & \ 1\ 89 \\
665 & \ \ \ 33\ 25 \\
 & \ \ \ 33\ 25
\end{array}$$

(b)

$$
\begin{array}{r|l}
 & \overset{\frown}{.01}\ \overset{\frown}{29}\ \overset{\frown}{96} \quad \underline{\sqrt{.114}} \quad \text{(Ans.)} \\
 & 1 \\
21 & \ \ 29 \\
 & \ \ 21 \\
224 & \quad 8\ 96 \\
 & \quad 8\ 96
\end{array}
$$

5. If the kitchen is square, all sides are of the same length. Therefore we can take the square root of the area to find the lengths of the sides.

$$\sqrt{196}=14$$

There are four sides so $4\times 14=56$ feet of border required.

6. (a)

$$
\begin{array}{r|l}
 & \overset{\frown}{20}.\overset{\frown}{79}\ \overset{\frown}{36} \quad \underline{\sqrt{4.56}} \quad \text{(Ans.)} \\
 & \underline{16} \\
85 & \ 4\ 79 \\
 & \ \underline{4\ 25} \\
906 & \quad 54\ 36 \\
 & \quad \underline{54\ 36}
\end{array}
$$

(b) Before we can find the square root of $17\frac{3}{8}$ we must change $\frac{3}{8}$ to a decimal.

$$\tfrac{3}{8}=3\div 8=.375$$

Then the number is 17.375.

Now we can find the square root of 17.375

$$
\begin{array}{r|l}
 & \overset{\frown}{17}.\overset{\frown}{37}\ \overset{\frown}{50}\ \overset{\frown}{00}\ \overset{\frown}{00} \quad \underline{\sqrt{4.1683}} \quad \text{(Ans.)} \\
 & \underline{16} \\
81 & \ 1\ 37 \\
 & \ \ \ \underline{81} \\
826 & \quad 56\ 50 \\
 & \quad \underline{49\ 56} \\
8328 & \qquad 6\ 94\ 00 \\
 & \qquad \underline{6\ 66\ 24} \\
83363 & \qquad\quad 27\ 76\ 00 \\
 & \qquad\quad \underline{25\ 00\ 89}
\end{array}
$$

7. This problem requires a little reasoning but is really very easy. If the whole piece of linoleum can be divided into four equal parts, then each part will have the same area as every other part. The area of the whole piece is 400, so each of the four equal parts will have an area of 100 square feet. We know the four equal parts are all square in shape. Therefore the sides of the four equal parts will all be the same length.

We can find the length of one side of one of the four squares because we know the area is 100 square feet.

$$\sqrt{100}=10$$

Thus the length of any side of the small squares is 10 feet.

The four squares taken together form the whole piece of linoleum and so the whole piece is $10+10+10+10=40$ feet long. The width of the squares is 10 feet. Therefore the whole piece is 40 feet long by 10 feet wide.

Check.

$$40 \times 10 = 400 \text{ square feet}$$

8. (**a**) First we have to square the fraction.

$$\left(\frac{2}{5}\right)^2 = \frac{2 \times 2}{5 \times 5} = \frac{4}{25}$$

Next multiply

$$\frac{\cancel{4}}{\cancel{25}} \times \frac{\cancel{5}}{\cancel{4}} = \frac{1}{5}$$

Next find the square root of $\frac{1}{5}$ or .20

$$\begin{array}{r|l}
 & .20\ 00\ 00 \quad \underline{/.447} \\
 & \underline{16} \\
84 & 4\ 00 \\
 & \underline{3\ 36} \\
887 & 64\ 00 \\
 & \underline{62\ 09}
\end{array}$$

PRACTICAL MATHEMATICS

Section 9

Lesson 1

For Step 1, recall all you learned about division, decimals, and fractions. For Step 2, learn the method of figuring and reducing ratios. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

RATIO

In our everyday life, whether in homes, factories, shops, drafting rooms, business offices, or on farms, we are continually making *comparisons* between things. In most cases these comparisons are in numerical terms where we compare one number to another. Typical examples are the comparisons we make between automobile prices, time, ages, miles, acres, income taxes, areas, sizes, units of measure, etc.

Suppose that you are comparing the prices of two automobiles, one of which is priced at \$2400 and the other at \$800. This comparison can be made in several ways. For example, you may say (1) that one automobile is priced \$1600 higher than the other; or (2) that one automobile is priced *three times as high as* the other. In the first case the comparison was made by subtraction (\$2400 − \$800 = \$1600). In the second case the comparison was made by division (\$2400 ÷ \$800 = 3). When you compare two numerical values by *division,* thus showing *how many times one contains the other,* or *is contained in* the other, you may not realize it, but you are actually using the *ratio method.*

Now let us see what *ratio* means.

Ratio. Ratio is the *relation* between two quantities or numbers, called **terms,** which are of the same kind. A ratio is found by dividing the first term by the second term.

The word "relation" can be thought of as meaning how much *larger* or *smaller* one term is than another term with which it is being compared.

Any two terms being compared must be of the same kind. We cannot compare picture frames to vacuum cleaners, or bolts to money, because they are not alike. We must always change our terms into like quantities before attempting to compare them. If they cannot be reduced to a common unit, no comparison is possible.

In all ratio calculations we have *two terms* given. For instance, in the previously mentioned automobile example we had two terms; namely, the \$2400 and the \$800. The \$2400 was mentioned first so it is the *first term.* The \$800 was mentioned second so it becomes the *second term.* In like manner all ratio examples or problems give the terms in this first and second order. As a further example, suppose we compare (find the relation between) 2 and 8. Here the 2 is mentioned first so it is the *first term,* and the 8 is mentioned second so it is the *second term.*

The definition for ratio says that a ratio (or relation) is found by dividing the first term by the second. In our automobile example we compared \$2400 and \$800. To find the ratio, according to definition, we must divide the first term by the second. The \$2400, as explained above, is the first term and the \$800 is the second. Then $\$2400 \div 800 = 3$. Therefore the ratio of \$2400 to \$800 is 3. In like manner the ratio of 2 to 8 is $2 \div 8 = .25$ or $\frac{1}{4}$. The following are calculated in the same way.

The ratio of 8 to 2 is $8 \div 2 = 4$
The ratio of 12 to 4 is $12 \div 4 = 3$
The ratio of 3 to 9 is $3 \div 9 = .33\frac{1}{3}$ or $\frac{1}{3}$
The ratio of 15 to 5 is $15 \div 5 = 3$
The ratio of 10 to 30 is $10 \div 30 = .33\frac{1}{3}$ or $\frac{1}{3}$
The ratio of 30 to 10 is $30 \div 10 = 3$
The ratio of 100 to 10 is $100 \div 10 = 10$
The ratio of 9 to 4 is $9 \div 4 = 2\frac{1}{4}$

In order to work to better advantage with such ratios as shown above, we use a symbol when writing ratios. The colon (:) is the symbol used. Thus instead of writing "the ratio of 15 to 5" we simply put the symbol between the 15 and 5 and write it "15 : 5." This means the same as writing out the words indicated. If we are *talking* about ratios we must use the words.

A ratio may also be expressed in the form of a fraction. For example, 8 : 2 can be written $\frac{8}{2}$ and has exactly the same meaning.

Or, $2:8$ can be written $\frac{2}{8}$. In your study of fractions you learned that $\frac{8}{2}$, for instance, means the same as $8 \div 2$ and that $\frac{2}{8}$ means the same as $2 \div 8$. We see that $\frac{8}{2}=4$ and that $\frac{2}{8}=\frac{1}{4}$. These ratios are the same as calculated previously for the same numbers, which proves that ratios may be expressed in the form of fractions, and in such form may be solved by the methods you learned for fractions.

Fig. 1 will help you to understand ratios. In the figure the letters A, B, C, etc., stand for the number of squares above them. Thus A stands for 9 squares, B stands for 8 squares, etc.

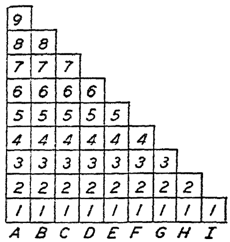


Fig. 1

Suppose we want to know the ratio of A to I or $A:I$. We know that A stands for 9 squares and that I stands for 1 square. The ratio can be written,

$$A:I \text{ or } \frac{A}{I} \text{ is } \frac{9 \text{ squares}}{1 \text{ square}}=9$$

Here we first wrote the ratio using the regular ratio symbol $(A:I)$. Then we wrote it as a fraction $\left(\frac{A}{I}\right)$. Next we wrote the same ratio substituting the number of squares that the letters stand for. We have already learned that the ratio of two numbers is found by dividing the first of the two terms by the second. Then $9 \div 1=9$ which is the ratio of $A:I$ or $\frac{A}{I}$. The following ratios illustrate the same principle.

$$A:G \text{ or } \frac{A}{G} \text{ is } \frac{9 \text{ squares}}{3 \text{ squares}}=3$$

$$D:F \text{ or } \frac{D}{F} \text{ is } \frac{6 \text{ squares}}{4 \text{ squares}}=1\frac{1}{2}$$

$$E:B \text{ or } \frac{E}{B} \text{ is } \frac{5 \text{ squares}}{8 \text{ squares}}=\frac{5}{8}$$

Any column of squares may thus be compared with another column, the same as we compared the prices of two automobiles. The comparisons or relations are the ratios and are found exactly as explained in the definition for Ratio and shown in previous examples.

Note this difference in ratios. A ratio having a first term *smaller* than the second term is similar to a *proper* fraction (Section 3) and the answer always will be less than 1; in other words it will be in decimal hundredths or a fraction—never a whole number. Example, $5:10=\frac{5}{10}=5\div10=.5$ or $\frac{1}{2}$. A ratio having a first term *larger* than the second term is similar to an *improper* fraction (Section 3) and the answer will be either a whole number or mixed number. Example 1. $10:5=\frac{10}{5}=10\div5=2$. Example 2. $11:5=\frac{11}{5}=11\div5=2.2$ or $2\frac{1}{5}$. This difference in ratios is pointed out to avoid future confusion.

Direct Ratio. When we divide a first term by a second term, to find a ratio, the ratio obtained is a *direct* ratio. For example, in the ratio $10:2$ the 10 is the first term and the 2 is the second term. Dividing the first term by the second term, we have $10\div2=5$. The 5 is a *direct* ratio. If the ratio is expressed in the form of a fraction, the ratio is the same, $\frac{10}{2}=5$.

Inverse Ratio. In the above ratio of $10:2$, if we divided the second term by the first term we would have an inverse ratio. Thus, $2\div10=.2$ or $\frac{1}{5}$. However, to avoid confusion, the following method of finding an inverse ratio is recommended.

To find an inverse ratio, (*a*) express the terms as a fraction (the first term above the second); (*b*) invert the fraction; (*c*) divide the numerator by the denominator as usual. For example, to find the *inverse ratio* of $10:2$. (*a*) Expressing the ratio $10:2$ as a fraction we have $\frac{10}{2}$; (*b*) inverting $\frac{10}{2}$ we have $\frac{2}{10}$; (*c*) dividing the numerator by the denominator (in other words, proceeding exactly as for a direct ratio) we have

$$\frac{2}{10}=2\div10=.2 \text{ or } \frac{1}{5}$$

which is the *inverse ratio* of $10:2$.

Note. When the numerator and denominator of a fraction are interchanged, the fraction is said to be inverted.

Simple Ratio. A simple ratio is the ratio between *two* terms.

Or, in other words, the ratio indicated by a first term and a second term is called a simple ratio. Thus all of the ratios you have studied thus far, such as \$2400 : \$800, 8 : 2, 2 : 16, etc., are simple ratios.

Fractions in Ratios. You have already learned that a ratio expression composed of whole numbers, such as 8 : 4 or 9 : 12, can be expressed in fraction form. Thus 8 : 4 can be expressed as $\frac{8}{4}$ and 9 : 12 as $\frac{9}{12}$. You know that 8 : 4 and $\frac{8}{4}$ both give the same ratio, namely, 2. In like manner 9 : 12 and $\frac{9}{12}$ both give $\frac{3}{4}$ as a ratio. You should remember that both of these ratios are composed of whole numbers which are expressed in the form of a fraction simply to indicate division.

Sometimes one term of a ratio is a fraction and we have such ratio expressions as $10 : \frac{1}{2}$ or $\frac{1}{5} : 30$. Here one of the original terms in each ratio is actually a fraction to start with. We can solve the $10 : \frac{1}{2}$, for example, by dividing 10 by $\frac{1}{2}$. Thus $10 \div \frac{1}{2} = \frac{10}{1} \div \frac{1}{2} = \frac{10}{1} \times \frac{2}{1} = \frac{20}{1} = 20$. Or, we can express the ratio $10 : \frac{1}{2}$ in the form of a fraction. To do so the first term (10) becomes the numerator and the second term ($\frac{1}{2}$) becomes the denominator. Then we have $\dfrac{10}{\frac{1}{2}}$. This is a complex fraction and its solution is also $10 \div \frac{1}{2} = 20$. If the ratio expression is $\frac{1}{5} : 30$ the ratio is found by dividing $\frac{1}{5}$ by $30 = \frac{1}{5} \div \frac{30}{1} = \frac{1}{5} \times \frac{1}{30} = \frac{1}{150}$. If we express this in terms of a fraction we have $\dfrac{\frac{1}{5}}{30}$. Here again we have a complex fraction and its solution is the same and gives a ratio of $\frac{1}{150}$.

Sometimes we have a ratio expression such as $\frac{3}{10} : \frac{3}{5}$. Here both of the original terms (first term and second term) are fractions. The solution is $\frac{3}{10} \div \frac{3}{5} = \dfrac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{10}}} \times \dfrac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{3}}} = \frac{1}{2}$. We can express this in fraction form and have

$$\dfrac{\frac{3}{10}}{\frac{3}{5}}$$

This is a complex fraction but it means to divide the numerator ($\frac{3}{10}$) (first term) by the denominator ($\frac{3}{5}$) (second term). The solution is the same as above.

Another way to solve a ratio expression such as $\frac{3}{10} : \frac{3}{5}$ would be to find a L.C.D. for the fractions and reduce the fractions to the terms of the L.C.D.

L.C.D. = 10, then

$$\frac{3}{10}=\frac{3}{10}$$
$$\frac{3}{5}=\frac{6}{10}$$

When the fractions have the same denominators, it is only necessary to express their numerators as a ratio. Thus we have $3:6$. Then $3\div 6=.5$ or $\frac{1}{2}$. Either of the above two methods of solution is correct to use where the first term and second term of a ratio are both fractions to start with.

Compound Ratio. If we have two ratio expressions by themselves and then multiply the first terms of the ratios together and multiply the second terms together and then state the relation of the first product to the second product (that is express the ratio between the two products) we have a compound ratio. This sounds complicated but it is really easy. Suppose we have the ratio expressions $8:4$ and $9:12$. Change these expressions to fraction form. Then we have $\frac{8}{4}$ and $\frac{9}{12}$. Multiply the two first terms (8 and 9) together and multiply the two second terms (4 and 12).

$$\text{(first terms)} \quad 8\times 9=72$$
$$\text{(second terms)} \quad 4\times 12=48$$

Now express a ratio (in fraction form) between 72 and 48. The first term is the numerator and the second term is the denominator. Then we have $\frac{72}{48}$. This can be reduced to lower terms and equals $\frac{3}{2}$. Therefore either $\frac{72}{48}$ or $\frac{3}{2}$ is a compound ratio of the *simple* ratio expressions $8:4$ and $9:12$.

Transformation of Ratios. The word transformation means a change of form. In ratio expressions, changes of form have various effects on the ratio. These effects we should know. Suppose we have the ratio, $12:2=6$. We can make certain changes in this expression and see what happens to the ratio.

Principle 1. *Multiplying the first term also multiplies the ratio.*

If the *first term* in the ratio $12:2=6$ is multiplied by any number, that change also multiplies the ratio by the same amount. Thus if we multiply 12 by 4 the ratio becomes $48:2=24$. You can see that the ratio has also been multiplied by 4.

Principle 2. *Dividing the first term also divides the ratio.*

If the first term in the ratio $12:2=6$ is divided by any number, that change also divides the ratio by the same amount. Thus if we divide 12 by 2 the ratio becomes $6:2=3$. You can see that the ratio has also been divided by 2.

Principle 3. *Multiplying the second term divides the ratio.*

If the second term of the ratio $12:2=6$ is multiplied by any number, that change divides the ratio an equal amount. Thus if we multiply 2 by 2 the ratio becomes $12:4=3$. You can see that the ratio has been divided by 2.

Principle 4. *Dividing the second term multiplies the ratio.*

If the second term of the ratio $12:2=6$ is divided by any number, that change multiplies the ratio an equal amount. Thus if we divide the second term by 2 the ratio becomes $12:1=12$. You can see that the ratio has been multiplied by 2.

Principle 5. *Multiplying or dividing both first term and second term by the same number does not alter the ratio.*

Thus in the ratio $12:2=6$, if we multiply the first term and second term by 2, we have $24:4=6$. If we divide both by 2, we have $6:1=6$.

General Principle. *A change in the first term brings about a like change in the ratio; but a change in the second term brings about an opposite change in the ratio.*

Application of Ratio. In the study of this subject and in its practical applications in everyday life, it often happens that problems arise in which we know some facts but are required to find a missing or unknown fact; for example, we may know the first term and second term and have to find the ratio; or we may know the second term and ratio and have to find the first term; or we may know the first term and ratio and have to find the second term. In the following, rules for each of the above conditions are given and explained.

Rule 1. *To find the ratio, divide the first term by the second term.*

This is the same rule as previously given in the definition of "ratio" in this lesson.

Rule 1 is easy to follow as we can easily decide which is the first term and which is the second because we know that the term mentioned first is the first term; the term mentioned second is the

second term. For example, if we are given the ratio expression 12 : 2, the 12 is the first term and the 2 is the second term. Therefore $12 \div 2 = 6$, which is the ratio.

Rule 2. *To find the first term, multiply the second term by the ratio.*

If we know the second term and the Ratio, we can always find the first term by this rule.

Suppose we know the second term is 2 and that the ratio is 6. Then

$$? : 2 = 6$$

Then, following **Rule 2,** we multiply $2 \times 6 = 12$ and know that 12 is the first term. We can prove this by using 12 as the first term and actually finding the ratio thus:

$$12 \div 2 = 6$$

Rule 3. *To find the second term, divide the first term by the ratio.*

If we know the first term and the ratio, we can always find the second term by this rule.

Suppose we know the first term is 12 and that the ratio is 6. Then

$$12 : ? = 6$$

Then, following **Rule 3,** we divide 12 by 6, or $12 \div 6 = 2$. The second term is therefore 2. We can prove this because if we know that 12 is the first term and that 2 is the second term, the ratio is $12 \div 2 = 6$.

Review

At this point in your study it is advisable to stop and make sure you thoroughly understand all that you have studied about Ratio. It is very easy to make such a review by looking over the following items. Read each one carefully, see if you can explain it to yourself, and, if necessary, go back and review the explanations concerning each item you do not feel sure about.

You should understand and be able to explain the following.

1. How to compare by division. (See page 1.)
2. What a ratio is. (See page 1.)
3. What the two parts of a ratio expression are. (See page 2.)
4. What must be true about things before they can be compared. (See page 2.)

5. How to tell which is the first and which is the second term. (See page 2).

6. How to solve a ratio. (See page 2).

7. What the ratio symbol is. (Page 2)

8. How to express a ratio in fraction form. (Page 3)

9. How to find the ratio of 1800 to 600. (Page 2)

10. What a direct ratio is. (Page 4)

11. What an inverse ratio is. (Page 4)

12. What a simple ratio is. (Page 4)

13. How to find a ratio when the first term and second term are fractions to start with. (Page 5)

14. What a compound ratio is. (Page 6)

15. What happens to the ratio when the first term is multiplied. (Page 6)

16. How to find the first term when you know the second term and ratio. (Page 8)

17. How to find the second term when the first term and ratio are known. (Page 8)

18. What happens to a ratio when the second term is multiplied.

19. What happens to a ratio when the first term is divided.

20. What happens to a ratio when the second term is divided.

If you can explain all of the above items correctly and thoroughly, you are ready to go on with this lesson. Spend all the time necessary to be sure of these items because you must fully understand these before you can understand what follows.

ILLUSTRATIVE EXAMPLES

1. What is the ratio of 9 to 3?

Solution. We want to find the *ratio* so **Rule 1** applies. To use **Rule 1** we must know which of the numbers (9 and 3) is the first term and which is the second term. This can be reasoned out very easily by remembering what you learned earlier in this lesson. We know that the first term is always the first of the two numbers given for any ratio expression. Therefore 9 is the first term and 3 becomes the second term.

Rule 1 says to divide the first term by the second term.

$$9 \div 3 = 3$$

The 3 is the required ratio.

2. Find the ratio of 20 to 5 by expressing the ratio in fraction form.

Solution. We know that when a ratio expression is put in the form of a fraction, the first term must be the numerator of the fraction, and the second term must be the denominator.

By the same reasoning process explained in the Solution to Example 1, we know that 20 is the first term and 5 the second term. Thus we have

$$20 \text{ to } 5 = 20 : 5 = \tfrac{20}{5}$$

Then, following **Rule 1,** we have

$$\tfrac{20}{5} = 20 \div 5 = 4$$

The 4 is the required ratio.

3. Find the ratio of 16 to 88.

Solution. By the same reasoning explained in the solution to Example 1, we know that 16 is the first term and that 88 is the second term.

Following **Rule 1,** we must divide the first term by the second term. In cases where the first term is a smaller number than the second term it is advisable to express the ratio in fraction form.

$$\begin{array}{ll} \text{(first term)} & \dfrac{16}{88} = \dfrac{\cancel{16}^{\,2}}{\cancel{88}_{\,11}} = \dfrac{2}{11} \\ \text{(second term)} & \end{array}$$

You can easily see that the $\frac{16}{88}$ was reduced to lowest terms by dividing both numerator and denominator by 8. The ratio is $\frac{2}{11}$.

If we were required to express this ratio as a decimal we would divide 2 by 11.

4. Find the ratio of 6 to $\frac{1}{3}$.

Solution. By the same reasoning as explained in the solution to Example 1, we know that 6 is the first term and that $\frac{1}{3}$ is the second term.

Where a fraction makes up either the first term or second term, it is advisable to express the ratio in fraction form. Thus we have,

$$\begin{array}{ll} \text{(first term)} & 6 \\ \hline \text{(second term)} & \frac{1}{3} \end{array}$$

Rule 1 is used. This is a complex fraction. The solution is as easy as any fraction example.

$$\frac{6}{\frac{1}{3}}=6\div\frac{1}{3}=\frac{6}{1}\div\frac{1}{3}=\frac{6}{1}\times\frac{3}{1}=\frac{18}{1}=18$$

The 18 is the required ratio.

5. Find the ratio of $\frac{1}{4}$ to 12.

Solution. The $\frac{1}{4}$ is the first term and 12 is the second term. Use **Rule 1.**

$$\frac{\frac{1}{4}}{12}=\frac{1}{4}\div 12=\frac{1}{4}\div\frac{12}{1}=\frac{1}{4}\times\frac{1}{12}=\frac{1}{48}$$

The $\frac{1}{48}$ is the required ratio.

6. Find the ratio of $\frac{3}{4}$ to $\frac{3}{5}$.

Solution. The $\frac{3}{4}$ is the first term and the $\frac{3}{5}$ is the second term. Using **Rule 1,** we have

$$\frac{\frac{3}{4}}{\frac{3}{5}}=\frac{3}{4}\div\frac{3}{5}=\frac{3}{4}\times\frac{5}{3}=\frac{15}{12}=1\frac{3}{12}=1\frac{1}{4}$$

The $1\frac{1}{4}$ is the required ratio.

7. Find the number whose ratio to 2 is 10.

Solution. Studying this example we can see that the 10 is the ratio. Also the 2 must be the second term because if we write out the example using a question mark in place of the missing number, we have the ratio of

$$?:2=10$$

Therefore the first term is the missing number as shown by the question mark.

We can use **Rule 2.** Multiplying the second term (2) by the ratio (10) we have 20, which is the first term.

Proof

$$20\div 2=10$$

8. The ratio of the length of a gate to its height is $5\frac{1}{2}$. The length is $19\frac{1}{2}$ feet. What is its height?

Solution. Studying this example we find that we know the ratio is $5\frac{1}{2}$ feet. Also we know that this $5\frac{1}{2}$ feet is the ratio of the length to the height of the gate. We can write,

The ratio of *length* to height is $5\frac{1}{2}$

or

$$\text{length} : \text{height} = 5\tfrac{1}{2}$$

The length is the first term and the height the second term because the length is given first and the height second. We know the length is $19\frac{1}{2}$ feet. Writing the above ratio again, we have

$$19\tfrac{1}{2} : \text{height} = 5\tfrac{1}{2}$$

Therefore the $19\frac{1}{2}$ is the first term and the missing height is the second term. To find second term, use **Rule 3.** We must divide the first term ($19\frac{1}{2}$) by the ratio ($5\frac{1}{2}$). Then

$$19\tfrac{1}{2} \div 5\tfrac{1}{2} = \tfrac{39}{2} \div \tfrac{11}{2} = \frac{39}{\cancel{2}_{1}} \times \frac{\cancel{2}^{1}}{11} = \tfrac{39}{11} = 3\tfrac{6}{11}$$

Therefore the second term is $3\frac{6}{11}$. In other words, the height is $3\frac{6}{11}$ feet.

Note. The above solution requires that you remember how to change mixed numbers to improper fractions (Section 3) and how to divide one fraction by another fraction (Section 4).

9. Find the ratio between 6 feet and 8 inches.

Solution. By the same reasoning as given in Example 1 we know that the first term is 6 feet and the second term is 8 inches. Therefore we must use **Rule 1.** *However,* you learned that both terms of a ratio expression must be in *like quantities.* Therefore we must change the feet to inches or the inches to feet. The 6 feet equals 6×12 or 72 inches. Then,

$$72 : 8 = \tfrac{72}{8} = 72 \div 8 = 9$$

The 9 is the ratio.

We could have changed the 8 inches to feet. We know that 8 inches is $\frac{8}{12}$ of a foot or $\frac{8}{12} = \frac{2}{3}$. Then,

$$6 : \tfrac{2}{3} = \frac{6}{\frac{2}{3}} = 6 \div \tfrac{2}{3} = \frac{\cancel{6}^{3}}{1} \times \frac{3}{\cancel{2}} = \tfrac{9}{1} = 9$$

10. The ratio of *A's* money to *B's* money is $\frac{1}{5}$. If *A's* money is \$6.25, how much money has *B?*

Solution. Studying the example, we can see that

$$\text{The ratio of } A \text{ to } B = \tfrac{1}{5}$$

or

$$A : B = \tfrac{1}{5}$$

We know that *A's* money is \$6.25. Therefore we can write,

$$\$6.25 : B = \tfrac{1}{5}$$

or

$$\$6.25 \div B = \tfrac{1}{5}$$

We know that *B's* money must be the second term, and so we use **Rule 3.** We must divide the first term (\$6.25) by the ratio ($\frac{1}{5}$).

$$3.25 \div \tfrac{1}{5} = \frac{6.25}{\frac{1}{5}} = \frac{6.25}{1} \div \tfrac{1}{5} = \frac{6.25}{1} \times \tfrac{5}{1} = \frac{31.25}{1} = \$31.25$$

The second term (*B's* money) is \$31.25.

Proof

$$\$6.25 \div \$31.25 = .2 \text{ or } \tfrac{1}{5}$$

11. What is the inverse ratio of 12 to 16?

Solution. If the word "inverse" were left out of the above example, we would, by the same reasoning as given in Example 1, readily see that 12 is the first term and 16 the second term. Expressing this as a fraction we have $\frac{12}{16}$. Inverting $\frac{12}{16}$ gives $\frac{16}{12}$. Dividing numerator by denominator (having simplified by cancellation) we have

$$\tfrac{16}{12} = \frac{\overset{4}{\cancel{16}}}{\underset{3}{\cancel{12}}} = \tfrac{4}{3} = 1\tfrac{1}{3}$$

The $1\frac{1}{3}$ is the inverse ratio of 12 to 16.

12. If the second term is $\frac{7}{8}$ and the ratio is $\frac{3}{4}$, what is the first term?

Solution. Use **Rule 2.**

$$\tfrac{7}{8} \times \tfrac{3}{4} = \tfrac{21}{32}$$

The $\frac{21}{32}$ is the first term.

Proof

$$\tfrac{21}{32} \div \tfrac{7}{8} = \tfrac{3}{4}$$

Be sure you can work all of the above examples without having to look at the solutions. You can do it if you study the examples carefully and if you thoroughly understand this lesson.

PRACTICE PROBLEMS

The following problems require that you thoroughly understand this lesson. Every problem can be worked by using the explanations in this lesson.

After you have worked the following problems, check your answers with the correct answers shown on page 33.

1. What is the ratio of 36 to 4?
2. What is the ratio of 7 to 49?
3. What is the ratio of 6 to $8\frac{1}{2}$?
4. What is the ratio of $6\frac{1}{2}$ to 78?
5. What is the ratio of 16 to 66?
6. What is the ratio of $\frac{5}{8}$ to $\frac{4}{16}$?
7. What is the ratio of $3\frac{1}{3}$ to $16\frac{2}{3}$?
8. What is the ratio of 3 gallons to 3 pints?
9. What is the inverse ratio of $\frac{2}{7}$ to $\frac{4}{9}$?
10. If the second term is 16 and the ratio $2\frac{2}{7}$, what is the first term?
11. What is the ratio of 60 to $\frac{1}{2}$?
12. What is the second term if the first term is 24 and the ratio is 3?
13. What is the ratio of 20 to .4?
14. The second term is $7\frac{4}{5}$, the first term is $3\frac{1}{4}$. What is the ratio?
15. The ratio of the length of a building to its width is $\frac{7}{5}$. If the width is 40 feet, what is the length?
16. Express the ratio of 6 hours to 25,000 seconds.
17. If the first term is 15 and the ratio is $\frac{4}{5}$, what is the second term?
18. If the second term is $3\frac{1}{4}$ and the ratio 7, what is the first term?
19. If the second term is \$6.12$\frac{1}{2}$ and the ratio is 25, what is the first term?
20. What is the inverse ratio of $5\frac{3}{4}$ to $17\frac{1}{4}$?

Lesson 2

For Step 1, recall Lesson 1, also simple multiplication and division. For Step 2, learn the meaning of proportion and how to find the different terms of proportions. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems. Be sure to work *all* Practice Problems.

PROPORTION

In Lesson 1 you learned how to compare two quantities or numbers by the division or ratio method. Then you learned how to solve typical examples and problems involving ratio.

In this lesson you will learn some new principles which are based on the ratio knowledge you gained from Lesson 1. These new principles will make it possible for you to solve problems which are a little more complicated than those you solved in Lesson 1.

Proportion. A proportion is an expression of equality between two ratios.

To illustrate this, suppose we think of the ratio expressions 8 : 4 and 12 : 6. If we use **Rule 1** and find the ratio for both expressions we see that 8 : 4 = 2 and 12 : 6 = 2. In other words both ratios are equal. If we write these expressions in such a manner as to show that one equals the other we have,

$$8 : 4 = 12 : 6$$

This expression forms a *proportion*. All proportions must be composed of two equal ratio expressions. As in the above expression, the equal sign (=) is used between the two ratio expressions and indicates that one is equal to the other.

Note: Sometimes four dots (::) are used in place of the equal sign, but the equal sign method is now being generally used.

The proportion 8 : 4 = 12 : 6 is read as follows:

8 is to 4 as 12 is to 6

Every proportion must have four terms because in all cases two ratio expressions form any complete proportion.

Proportion can also be expressed in the fraction form. In Lesson 1 you learned that a ratio expression, when written in fraction form, has the first term as the numerator and the second term as the denominator. Thus the ratio expressions 8 : 4 and 12 : 6 are written $\frac{8}{4}$ and $\frac{12}{6}$. As a proportion this would be written $\frac{8}{4} = \frac{12}{6}$.

In ratio we called the quantities or numbers first and second terms. This was an easy way of designating the terms. In proportion we must learn two new names in order to work to better advantage with the two terms composing each ratio.

extremes

$$8 : 4 = 12 : 6$$

means

The above diagram illustrates how the names "means" and "ex-

tremes" are applied to the two sets of terms in a proportion. Thus it can be seen that the two inner numbers (4 and 12) are the *means* and that the two outer numbers (8 and 6) are the *extremes.*

In the proportion $3:4=6:8$

3 and 8 are the extremes.
4 and 6 are the means.

In the proportion $8:9=24:27$

8 and 27 are the extremes.
9 and 24 are the means.

In the proportion $\frac{5}{6}=\frac{15}{18}$

5 and 18 are the extremes.
6 and 15 are the means.

In the proportion $\frac{4}{2}=\frac{60}{30}$

4 and 30 are the extremes.
2 and 60 are the means.

In Lesson 1 you learned that if you know any two of the three numbers of a ratio you can find the missing third number. This process was based on a series of rules which were explained.

A somewhat similar situation exists in proportion because, if we know three of the four numbers we can easily find the missing fourth number. Several rules are necessary. These are explained in the following:

Rule 4. *The product of the means is equal to the product of the extremes.*

Take, for example, the proportion

$$8:4=12:6$$

The product of the means is $4\times12=48$. The product of the extremes is $8\times6=48$. The ratios are equal and thus the product of the means equals the product of the extremes. *In any proportion this is true.*

If the proportion is expressed in fraction form we have $\frac{8}{4}=\frac{12}{6}$. **Rule 4** still holds true because 4 and 12 are the means $(4\times12=48)$ and 8 and 6 are the extremes $(8\times6=48)$.

You must remember that no proportion expression is a true proportion unless the two ratios are equal, or, in other words, unless **Rule 4** holds true.

Rule 5. *The product of the extremes divided by either mean gives the other mean as quotient.*

Take, for example, the proportion

$$8 : 4 = 12 : 6$$

Suppose the 4 (one of the means) were missing. We would have

$$8 : ? = 12 : 6$$

In order to use **Rule 5** we must find the product of the extremes and divide this product by the known mean.

$$8 \times 6 = 48 \text{ (product of extremes)}$$
$$48 \div 12 = 4$$

Therefore the missing mean is 4. We know this is true because with 4 as the missing mean the product of the means equals the product of the extremes.

Rule 6. *The product of the means divided by either extreme gives the other extreme as quotient.*

Take, for example, the proportion

$$8 : 4 = 12 : 6$$

Suppose the 6 (one of the extremes) were missing. Then the expression would be,

$$8 : 4 = 12 : ?$$

In order to use **Rule 6** we must find the product of the means and divide the product by the known extreme.

$$4 \times 12 = 48 \text{ (product of means)}$$
$$48 \div 8 = 6$$

Therefore the other or missing extreme is 6. This is true because with the 6 as the missing extreme the product of the means is equal to the product of the extremes.

With **Rules 4, 5, and 6** we can find any missing number in a true proportion. You should memorize these rules and make sure you fully understand how to use them.

ILLUSTRATIVE EXAMPLES

1. Find the missing number in $3 : 4 = 12 : ?$

Solution. By inspection we see that the missing or unknown number (indicated by question mark) is one of the extremes. Therefore **Rule 6** must be used.

The product of the means is $4 \times 12 = 48$. Then $48 \div 3 = 16$. Thus the 16 is the missing extreme.

Proof

$$3 : 4 = 12 : 16$$
$$4 \times 12 = 48 \text{ (product of means)}$$
$$3 \times 16 = 48 \text{ (product of extremes)}$$

The product of the means equals the product of the extremes so our answer is correct.

2. Find the missing number in $5 : .3 = ? : 6$

Solution. By inspection we see that the missing number is one of the means. Therefore we must use **Rule 5.**

The product of the extremes is $5 \times 6 = 30$. Then $30 \div .3 = 100$. Thus the 100 is the missing mean.

Proof

$$5 : .3 = 100 : 6$$
$$5 \times 6 = 30 \text{ (product of extremes)}$$
$$100 \times .3 = 30 \text{ (product of means)}$$

3. Find the missing number in $? : 120 = 8 : 192$

Solution. By inspection we see that the missing number is one of the extremes. Therefore we must use **Rule 6.**

$$120 \times 8 = 960$$
$$960 \div 192 = 5$$

Thus the missing extreme is 5.

Proof

$$5 : 120 = 8 : 192$$
$$120 \times 8 = 960$$
$$5 \times 192 = 960$$

4. Find the missing number in $\dfrac{36}{?} = \dfrac{12}{17}$

Solution. By inspection we see that the missing number is one of the means. This is easy to tell because if this proportion were changed from fraction to regular form it would be $36 : ? = 12 : 17$. **Rule 5** must be used.

$$36 \times 17 = 612$$
$$612 \div 12 = 51$$

Thus the missing mean is 51.

Proof

$$\frac{36}{51}=\frac{12}{17}$$

$$51\times 12=612$$

$$36\times 17=612$$

PRACTICE PROBLEMS

After you have worked the following problems, check your answers by the correct answers shown on page 33.

Supply the missing number in the following proportions:

1. $17:?=51:54$
2. $\frac{19}{?}=\frac{209}{143}$
3. $?:3.9=40\quad 78$
4. $\frac{\text{[illegible]}}{.5}=\frac{?}{20}$
5. $\frac{35}{7}=\frac{?}{91}$
6. $48:20=?:50$
7. $?:300=20:100$
8. $1:?=7:84$
9. $48\text{ yd.}:?=\$67.25:\$2$
10. $12:5=?:40$
11. $16:24=?:15$
12. $3\frac{1}{2}:4\frac{2}{3}=9\frac{3}{7}:?$
13. $41\frac{1}{3}:?=196\frac{1}{3}:232\frac{1}{2}$
14. $?:9.75=13.50:10.40$
15. $2.76:3.45=2.28:?$

Lesson 3

For Step 1, recall the two previous lessons on ratio and proportion. For Step 2, learn how to solve problems involving simple proportion. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

PROBLEMS INVOLVING PROPORTION

When you were solving problems in Lesson 2, the various numbers of a proportion were all shown in their proper positions and it was only necessary for you to apply the correct rule in order to solve it. In this lesson we will go one step further and learn how to place the numbers of a proportion after studying the written statement of the problem to be solved.

You have already learned that a proportion has four terms, that the inner two are called means, and that the outer two are called extremes. Be sure to review Lesson 2 if this is not clear to you.

Any proportion has two ratios. The ratio which comes first is called the first ratio, and the one which comes second is the sec-

ond ratio. A ratio has a first and a second term. Therefore we speak of the first term of the first ratio; the second term of the second ratio, etc. In the proportion $8:2=12:3$, for example, the 8 is the first term of the first ratio, the 2 is the second term of the first ratio, the 12 is the first term of the second ratio, and the 3 is the second term of the second ratio.

You must remember that *in a proportion where the first term of the first ratio is larger than the second term of the same ratio, the first term of the second ratio must be larger than the second term of that ratio.* Thus in the proportion $8:2=12:3$ the 8 and 12 are larger than the 2 and 3.

If the first term of the first ratio is smaller than the second term of the same ratio, then the first term of the second ratio must be smaller than the second term of the same ratio. Thus in the ratio $12:20=15:25$ the 12 and 15 are smaller than the 20 and 25.

From now on the problems will be presented so that they merely give a statement of certain conditions and an indication of what is required in the way of an answer. For example, read over the statement of the first Illustrative Example which follows this discussion. Here you can see what you have to start with in the average problem. To solve such problems it is necessary to determine which of the quantities or numbers is the first term in the first ratio, which is the second term in the first ratio, etc.

In all proportion problems three of the four numbers which become terms of the two ratios are given in the statement of the problem. In Illustrative Example 1 notice that there are three numbers. The missing number is what we solve for. The placing of these numbers will be much easier if we keep in mind that the first two terms in a proportion form a ratio; that the last two terms form a ratio; and that in a ratio only similar things can be used. Thus the first ratio in a proportion for Illustrative Example 1 must contain only such numbers as have the same meaning. If we put the 16 and 11 (both of which mean feet) in the first ratio, we will be comparing one measurement in feet to another measurement in feet, which is correct. The second ratio must also have numbers representing the same thing. We have one number (80) which represents rivets, and the answer we want to find will represent rivets. Therefore the 80 and the unknown answer will be in the second ratio.

In order to have a definite plan for deciding where to place the three numbers given in any problem, *we will always make the unknown answer the second term in the second ratio.* This unknown answer will be indicated by the small letter x. The actual placing of the three given numbers is explained in the following Illustrative Examples. If you follow the reasoning carefully, you will not have any trouble.

Once the placing of the three numbers has been accomplished, the problems are as easy as the problems in Lesson 2. As we will always indicate the answer by an x placed as the second term in the second ratio, it will be one of the extremes and **Rule 6** will be used for solving.

ILLUSTRATIVE EXAMPLES

1. A metal joint 16 feet long requires 80 rivets. How many rivets are required for a joint 11 feet long?

Solution

Instruction	*Operation*
Step 1 The first thing to do is to write the ratio symbols and the equal sign between what will be the ratios. The x is put in as the second term of the second ratio, as explained in the preceding discussion.	**Step 1** $= \; : x$
Step 2 We know our answer must represent a number of rivets because that is what the question asks for. So the x represents an unknown number of rivets. To complete the second ratio (the one containing the x) we must select another number that represents rivets because both terms of a ratio must be of the same kind or represent the same things. Thus we can put 80 in this ratio because it represents rivets. Now the second ratio is complete.	**Step 2** $: \; = 80 : x$

Step 3

In Step 2 the ratio expression $80 : x$ represents rivets. We can easily reason out that fewer rivets will be necessary for the 11-foot joint than for the 16-foot joint. We know that 80 rivets are for the 16-foot joint and that x is the number of rivets in the 11-foot joint. Therefore in the ratio expression $80 : x$ the 80 is the larger number of the two terms making up the ratio expression. In the first ratio, the larger number will have to come first too, as previously explained. The first ratio will be a comparison between the lengths of the joints. Both the 16 and 11 represent feet so they will make up the first ratio, with the larger number (16) coming first. Now we have the proportion expressed.

Step 3

$$16 : 11 = 80 : x$$

Step 4

The unknown value (x) is an extreme, so we must use **Rule 6** to solve the proportion. We must divide the product of the means by the known extreme.

Two methods of calculation are shown. One is the long method and the other shows how cancellation may be used to shorten the process. In the cancellation method, 4 was used as a divisor.

The answer is 55 rivets.

Step 4

(long method)

$$11 \times 80 = 880 \text{ (product of means)}$$
$$880 \div 16 = 55 \text{ (Answer)}$$

(cancellation method)

$$\frac{11 \times \overset{\overset{5}{\cancel{20}}}{\cancel{80}}}{\underset{\underset{1}{\cancel{4}}}{\cancel{16}}} = \frac{55}{1} = 55$$

Proof

$$16 : 11 = 80 : 55$$
$$11 \times 80 = 880 \text{ (product of means)}$$
$$16 \times 55 = 880 \text{ (product of extremes)}$$

The solution of the proportion is correct because the product of the means equals the product of the extremes.

2. If 15 men can build a wall 12 feet high, how many men will be required to build one 20 feet high?

Solution

Instruction	*Operation*
Step 1 Start out as in Example 1 by writing the ratio symbols and the equal sign. The x is put in as the second term of the second ratio as previously explained.	**Step 1** $= : x$
Step 2 The answer x will be a number of men, so the other term of the second ratio must be a number of men too, since only similar things can be used in a ratio. The given number of men is 15, so the first term of the second ratio is 15.	**Step 2** $: = 15 : x$
Step 3 To place the terms in the first ratio: The number of men required to build 20 feet will be greater than the number required for 12 feet, so x is greater than 15. Therefore because the greater is the second term of the second ratio, the greater of the two remaining numbers must be the second term of the first ratio, thus, $12 : 20$.	**Step 3** $12 : 20 = 15 : x$
Step 4 Find x (the second term in the second ratio) by using **Rule 6.** $(20 \times 15) \div 12 = 25$ men. Answer.	**Step 4** $\dfrac{\overset{5}{\cancel{20}} \times \overset{5}{\cancel{15}}}{\underset{4}{\cancel{12}}} = 25$

Proof

$$12 : 20 = 15 : 25$$

$$20 \times 15 = 300 \text{ (product of means)}$$

$$12 \times 25 = 300 \text{ (product of extremes)}$$

The product of the means equals the product of the extremes.

3. If a mountain road rises 5 feet in every 40 feet of distance, how much will it rise in 100 feet of distance?

Solution

Instruction	*Operation*
Step 1 Place the signs as in the previous problems, and the x as a symbol for the unknown.	**Step 1** $= : x$
Step 2 The answer is to be a number of feet of "rise," so the given number of feet of rise must be put in the same ratio with the x.	**Step 2** $: = 5 : x$
Step 3 Now, to place the terms in the first ratio: The rise in 100 feet will be greater than the rise in 40 feet, so x is greater than 5. Therefore, the greater "distance" in feet will be placed as the second term of the first ratio thus, 40 : 100. Remember that if the first term of one of the ratios is smaller than the second term, then the first term of the other ratio will have to be smaller than its second term. You learned this in a previous part of this lesson.	**Step 3** $40 : 100 = 5 : x$
Step 4 Find the second term of the second ratio, x, using **Rule 6.** $(100 \times 5) \div 40$. There will be a rise of $12\frac{1}{2}$ feet. Answer.	**Step 4** $\frac{100 \times \cancel{5}}{4\cancel{0}} = \frac{100}{8} = 12\frac{4}{8} = 12\frac{1}{2}$

This can be proved as shown for Examples 1 and 2 because the product of the means equals the product of the extremes.

4. If a pole 3 feet 6 inches in height casts a shadow 18 inches long, what is the height of a steeple that casts a shadow 40 feet long at the same time?

Solution

Instruction

Step 1

Place the signs in position for the proportion, placing x as the second term in the second ratio.

Operation

Step 1

$$= : x$$

Notice that one of the lengths mentioned in the problem is given in feet and inches, another in inches and another in feet. Remembering that only similar things can be placed in the same ratio, you will know that all these must be changed to either feet or inches. We will change to inches so as to avoid having to use fractions. We change feet to inches by multiplying the number of feet by 12 inches.

Step 2

Height of pole, 3 ft. 6 in. = 42 in.
Length of first shadow = 18 in.
Length of second shadow, 40 ft. = 480 in.
The required answer is the *height* of the steeple, so the *height* of the pole must be put in the same ratio—comparing similar things.

Step 2

$$: = 42 : x$$

Step 3

The longer the shadow, the greater the height of the object casting the shadow, so x is greater than 42, since the 40-foot shadow is longer than the 18-inch shadow. Therefore in the first ratio place the 480 as the second term.

Step 3

$$18 : 480 = 42 : x$$

Step 4

Find second term in second ratio (x). By **Rule 6** $(480 \times 42) \div 18$ will be required height. 1120 inches or $93\frac{1}{3}$ feet. Ans.

Step 4

$$\frac{\overset{160}{\cancel{480}} \times \overset{7}{\cancel{42}}}{\underset{3}{\cancel{18}}} = 1120$$

5. If it requires 18 hours to saw 10,000 feet of lumber, using a 20 horsepower engine, what horsepower will be required to saw the same amount in 10 hours?

Solution

Instruction	*Operation*
Step 1 (The 10,000 feet does not enter into the calculations as it remains the same under both conditions.) Arrange the signs for the proportion, placing x as the last term in the second ratio.	**Step 1** $=\ :x$
Step 2 The x is a number of horsepower, so the other number expressing horsepower is the first term of that ratio.	**Step 2** $:\ =20:x$
Step 3 Under the conditions of this problem, the time is decreased from 18 hours to 10 hours, so the horsepower will have to be increased to do the same amount of work, therefore, x is greater than 20. So put the larger remaining number, 18, as the second term of the first ratio; thus, 10 : 18.	**Step 3** $10:18=20:x$
Step 4 Find the second term of the second ratio (x) by **Rule 6.** $(18\times 20)\div 10=36$ hp. Ans.	**Step 4** $\dfrac{18\times \overset{2}{\cancel{20}}}{\cancel{10}}=36$

Summary. In Examples 1 to 4 you found that the required quantity or number was larger or smaller as the given conditions became larger or smaller. In Example 1, the longer the joint, the larger the number of rivets required; in Example 2, the higher the wall, the greater the number of men required; in Example 3, the longer the road, the higher the rise; and in Example 4, the longer the shadow, the greater the height. In Example 5, as the time is *lessened* from 18 hours to 10 hours, the horsepower is to be *increased*. However, note that in Example 5 we reasoned that x is the larger term in the second ratio, so we used the larger term last in the first ratio as well.

As you studied the five examples, you may have noticed that, in order to solve examples in proportion, a definite procedure was followed in each case. The following will help you to remember how the Illustrative Examples were solved and help in working the Practice Problems.

Procedure for Solving Proportion Problems

1. Denote the required quantity by an x. Place this x as the second term of the second ratio.

2. Use the quantity or number with which x is compared as the first term of the second ratio.

3. Find out, by a careful study of the problem, whether the answer (x) should be greater or less than the first term of the second ratio.

4. Write the first and second terms of the first ratio in place, remembering that if the first term of the *second* ratio is larger than its second term, then the first term of the *first* ratio should be larger than its second term, etc. This was explained previously.

5. Use **Rule 6** to solve the proportion.

Note: Before going ahead from this point you should make sure that you can work all of the Illustrative Examples without referring to the solutions.

PRACTICE PROBLEMS

After you have worked the following problems, compare your answers with the correct answers shown on page 33.

1. If 5 tons of coal cost $30, what will 3 tons cost?

2. If 12 yards of cloth cost $48, what will 4 yards cost?

3. What will 11 pounds of a product cost, if 3 pounds 12 ounces cost $3.50?

4. If 3 quarts of milk can be bought for 30 cents, how many quarts can be bought for 90 cents?

5. If you travel 90 miles in 3 hours, how far would you travel, at the same speed, in 7 hours?

6. How many hours would it take you to go 180 miles if you went 120 miles in 6 hours?

7. If you spent $16 during the first 4 days of a vacation, how long a vacation could you have for $40 at the same rate of expense?

8. If the taxes on a house of an assessed value of $6,000 were $168, how much would the taxes be on a house, at the same rate, which was assessed at $9,000?

9. If you used 15 gallons of gasoline in driving 255 miles, how much would you use for a trip of 425 miles?

10. If a printing press printed 675 pages of material in 15 minutes, how long would be required to print 2,700 pages?

Lesson 4

For Step 1, review Lessons 1 and 2. For Step 2, learn the meaning of Compound Proportion and how to apply it to problems. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Spend ample time on this lesson because it requires thorough studying.

COMPOUND PROPORTION

In Lessons 2 and 3 you studied what we might call "simple" proportion. In those lessons the proportions you worked with were made up of four terms, each of which represented *one* thing such as men, joints, roads, hours, etc.

In this lesson you will study a type of proportion where some of the terms are *products* of *two* things. In other words some of the terms have two parts which have to be multiplied together. In proportions where some of the terms are products, we call the expression a "compound" proportion.

You will learn all the details about compound proportion as you study this lesson. However, for a first comparison between simple and compound proportions and as illustration for the preceding two paragraphs, look at the proportion shown in Step 3, in the first Illustrative Example in Lesson 3, and the proportion shown in Step 2 of the first Illustrative Example in this lesson. This comparison will show you how some of the terms in compound proportion differ from those in simple proportion. You can readily see, for instance, that some of the terms in compound proportion are *products*.

In compound proportion we must learn two new names which we shall use to advantage in the placing of the terms to form compound proportions. Every question in any proportion, and especially in a compound proportion, may be considered as a comparison between two *causes* and two *effects*.

Causes. Causes may be thought of as action on the part of, for example, the consumer, men, animals, time, distance, weight, goods bought or sold, etc.

Effects. Effects may be thought of as the results of action, the thing produced or consumed, expense, money paid, etc.

To better understand what *causes* and *effects* mean, we will study the following statement and the proportion representing it.

If 5 tons of coal cost \$30, then 3 tons will cost \$18.

or

$$5 : 3 = 30 : 18$$

Here the 5 and 3 are clearly *causes* because they are goods (coal) purchased or they are the act (action) of buying coal. The 30 and 18 are clearly *effects* because these numbers represent money paid, or they are the results of the action of buying coal.

Using the words *causes* and *effects* we write the proportion.

$$5 : 3 = 30 : 18$$

or

1st cause : 2nd cause = 1st effect : 2nd effect

Here the 5 is the 1st cause, the 3 is the 2nd cause, the 30 is the 1st effect, and the 18 is the 2nd effect.

In solving proportion problems we have learned to put the unknown value (answer required) in the proportion as the second term of the second ratio. Sometimes in solving compound proportion problems, the unknown value (answer required) is a cause and sometimes an effect. If it is an *effect* we write out the proportion like this

1st cause : 2nd cause = 1st effect : 2nd effect

Thus we can put the x, which represents the unknown answer, in its proper place as you learned in Lesson 3. If the unknown value (answer required) is a *cause* we write the proportion like this

1st effect : 2nd effect = 1st cause : 2nd cause

Thus we can put the x, which represents the unknown answer, in its proper place as you did in Lesson 3.

In solving compound proportion problems the placing of the terms is the same as you learned in Lesson 3, except that in either the *causes* or *effects* the terms will be made up of products. This can best be illustrated by examples.

ILLUSTRATIVE EXAMPLES

1. If 18 men can build a wall 42 rods long in 16 days, how many men can build a wall 28 rods long in 8 days?

Solution

Instruction

Step 1

In this problem we can easily see that "men" and "days" are the *causes* and that "rods" are the effects. The men create action over a period of time, and the rods of fence built are an effect, or purely a result of action. The answer required is in terms of "number of men." We always place the x, which represents the answer, as the second term of the second ratio. Effects make up the first ratio and causes (men) the second.

Step 2

Now we want to place the various terms and the reasoning is much like you learned in Lesson 3. We know that the second term in the second ratio is "men." Therefore the first term must be "men" too because ratios only include like terms. Thus the first term in the second ratio must be "18 men." But we know that the *causes* are "men" *and* "days" so the terms of the second ratio must be a product of men and days. Or, the 1st cause is the product of 18 and 16 and the 2nd cause is the product of 8 and x.

The first term of the second ratio is larger than the second term. We know this because the number of men required in the answer work only 8 days and build less wall.

The effects are 42 and 28. The 42 must come first, in the first ratio, because the first term in the second ratio is the larger in the second ratio.

Operation

Step 1

$$= : x$$

This proportion written in words using effects and causes is

$$\text{1st effect} : \text{2nd effect} = \text{1st cause} : \text{2nd cause } (x)$$

Step 2

$$42 : 28 = 18 \times 16 : 8 \times x$$

or

$$42 : 28 = 288 : 8 \times x$$

Step 3

Use **Rule 6.** We find the product of the means first. Then we divide this product by the other extreme (42). The 192 is the value of $8 \times x$. If $8 \times x = 192$, then $x = 192 \div 8$. The answer is 24 men. This can be proved as shown in Lesson 3.

Step 3

$$28 \times 288 = 8064$$
$$8064 \div 42 = 192$$
$$192 \div 8 = 24$$

2. If it takes 4 days for 100 workmen to erect 1 story of a building, how many stories can 50 men erect in 64 days?

Solution

Instruction

Step 1

Our answer is work accomplished, or an effect. The proportion is arranged as shown. Here we have the 1st and 2nd causes in the first ratio because the x (answer) is an effect and it must always be the second term of the second ratio. This was explained previously.

Step 2

The first cause is 100×4. The second cause is 50×64. The first effect is 1. The second effect is x. Put these values in our proportion. You will be able to reason out this step by applying the same type of reasoning given for Example 1.

Step 3

Find x (the unknown extreme) by **Rule 6.** $(3200 \times 1) \div 400 = 8$ stories. (Answer)

Operation

Step 1

$$: = : x$$

$$\text{1st cause} : \text{2nd cause} = \text{1st effect} : \text{2nd effect } (x)$$

Step 2

$$(100 \times 4) : (50 \times 64) = 1 : x$$
$$\text{or } 400 : 3200 = 1 : x$$

Step 3

$$\frac{32\cancel{00} \times 1}{4\cancel{00}} = 8$$

PRACTICE PROBLEMS

After you have worked the following problems, compare your answers with the correct answers shown on page 33.

1. If 15 men can excavate 240 cubic feet in 4 days, how long will it take 50 men to excavate 1000 cubic feet?

2. If a man travels 120 miles in 3 days when the days are 12 hours long, how many days of 10 hours each will he require to travel 360 miles?

3. If 16 horses consume 128 bushels of oats in 50 days, how many bushels will 5 horses consume in 90 days?

4. If 6 laborers dig a ditch 34 yards long in 10 days, how many yards can 20 laborers dig in 15 days?

ANSWERS TO PRACTICE PROBLEMS

Lesson 1, Page 14

1. 9. **2.** $\frac{1}{7}$. **3.** $\frac{12}{17}$. **4.** $\frac{1}{12}$. **5.** $\frac{8}{33}$. **6.** $2\frac{1}{2}$. **7.** $\frac{1}{5}$. **8.** 8. **9.** $1\frac{5}{9}$. **10.** $36\frac{4}{7}$. **11.** 120. **12.** 8. **13.** 50. **14.** $\frac{5}{12}$. **15.** 56 feet. **16.** $\frac{108}{125}$. **17.** $18\frac{3}{4}$. **18.** $22\frac{3}{4}$. **19.** $153.125. **20.** 3.

Lesson 2, Page 19

1. 18. **2.** 13. **3.** 2. **4.** 240. **5.** 455. **6.** 120. **7.** 60. **8.** 12. **9.** 144 yards. **10.** 96. **11.** 10. **12.** $12\frac{4}{7}$. **13.** $48\frac{18}{19}$. **14.** $12.65\frac{5}{8}$. **15.** 2.85.

Lesson 3, Page 27

1. $18. **2.** $16. **3.** $10.26+. **4.** 9 quarts. **5.** 210 miles. **6.** 9 hours. **7.** 10 days. **8.** $252. **9.** 25 gallons. **10.** 60 minutes.

Lesson 4, Page 32

1. 5 days. **2.** $10\frac{4}{5}$ days. **3.** 72 bushels. **4.** 170 yards.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or final examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on page 36.

If you miss more than two of the problems it means you should review the whole section very carefully.

Do not try this trial examination until you have worked every practice problem in this Section.

Do not start the final examination until you have completed this trial examination.

1. In a ratio expression, 24 is the first term and $\frac{3}{8}$ is the second. Find the ratio.

2. If the second term is 3 and the ratio is 6, find the first term.

3. Find the missing term in the following proportion. $4\frac{1}{4} : 38\frac{1}{4} = ? : 76\frac{1}{2}$.

4. What is the inverse ratio of .20 to .5?

5. If 5 tons of coal cost $30, what will 3 tons cost?

6. If 15 barrels of flour cost $90, how many barrels can be bought for $30?

7. An insolvent debtor fails for $7560, of which he is able to pay only $3100; how much will *A* receive whose claim is $756?

8. If $480 gains $84 interest in 30 months, what sum will gain $21 in 15 months?

9. If 5 men can reap 52.2 acres in 6 days, how many men will reap 417.6 acres in 12 days?

10. If you *divide* the first term of a ratio expression what happens to the ratio?

FINAL EXAMINATION

1. The ratio is $\frac{4}{5}$ and the first term is 15. Find the second term.

2. In a ratio expression, $\frac{5}{16}$ is the first term and 9 is the second term. Find the ratio.

3. Supply the missing term in each of the following proportions.

(a) $23 : 15 = 46 : ?$

(b) $4 : 36 = ? : 81$

4. Supply the missing terms in the following proportions.

(a) $$\frac{84}{?} = \frac{60}{5}$$

(b) $$\frac{44}{0.8} = \frac{55}{?}$$

5. (a) What is the inverse ratio of 48 to 12?

(b) What is the inverse ratio of $\frac{16}{32}$?

6. Two numbers are in the ratio of $\frac{3}{4}$ to $\frac{9}{8}$ and the lesser number is \$164.50. Find the greater number.

7. If 15 masons build a brick wall in $9\frac{3}{5}$ days, how long will 35 masons need to do the same work?

8. If a flagstaff 105 feet high casts a shadow 241 feet 6 inches long, how tall is a building that casts, at the same time, a shadow 517.5 feet long?

9. If a train traveling 40 miles an hour can reach a division on the line in $3\frac{1}{2}$ hours, how soon can a train going 56 miles an hour reach it?

10. If 504 bricks 8 inches long and 4 inches wide are required for a walk 28 feet long and 4 feet wide, how many bricks of the same size will be needed for a walk 124 feet long and 5 feet wide?

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. This ratio expression would be written

$$24 :$$

We know from **Rule 1** that the first term (24) divided by the second term ($\frac{3}{8}$) equals the ratio.

Thus $$24 \div \tfrac{3}{8} = \tfrac{24}{1} \div \tfrac{3}{8} = \frac{\overset{8}{\cancel{24}}}{1} \times \frac{8}{\cancel{3}} = \tfrac{64}{1} = 64 \text{ (Ans.)}$$

2. Here we can write the problem like this:

$$? : 3 = 6$$

The 3 is in the position of the second term, and 6 is shown as the ratio. The first term is missing. Use **Rule 2** which says to find the first term multiply the second term by the ratio.

$$\text{Thus } 6 \times 3 = 18 \text{ (Ans.)}$$

Proof

$$18 \div 3 = 6$$

3. In this proportion one of the means is missing. **Rule 5** must be used here. To use **Rule 5** we must divide the product of the extremes by one of the means. The product of the extremes is $4\frac{1}{4} \times 76\frac{1}{2} = \frac{17}{4} \times \frac{153}{2} = \dfrac{2601}{8}$.

Then

$$\frac{2601}{8} \div \frac{153}{4} =$$

$$\frac{\overset{17}{}}{\underset{2}{\cancel{8}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{153}}} = \tfrac{17}{2} = 8\tfrac{1}{2} \text{ (Ans.)}$$

4. You learned in the study of the lessons that the inverse ratio is found by expressing the ratio as a fraction, inverting, and dividing the numerator by the denominator. Thus to find the inverse ratio of .20 to .5

$$\frac{.20}{.5} \text{ inverted} = \frac{.5}{.20} = .5 \div .20 = 2.5$$

5. We start the solution of such a proportion by placing x, the unknown answer, as the second term of the second ratio. Then we have

$$: = : x$$

To fill in the first term of the second ratio we know that the quantity must be of the same kind as x. The x (answer) will be dollars. So we place \$30 as the first term in the second ratio. Now we have

$$: = \$30 : x$$

We must complete the first ratio using 5 and 3. Both of these represent quantities of coal in tons. The answer will be less than $30 because we are finding the cost of only 3 tons whereas 5 tons cost $30. Thus the first term of the second ratio is larger than the second term. Then the first term in the first ratio must be larger than the second term in the first ratio. Thus we have

$$5 : 3 = \$30 : x$$

We can easily see that one of the extremes must be calculated. Therefore we must use **Rule 6.**

$$3 \times \$30 = \$90$$
$$\$90 \div 5 = \$18$$

Then $18 is the answer.

Proof

$5 : 3 = 30 : 18$ (the proportion)
$3 \times 30 = 90$ (product of means)
$5 \times 18 = 90$ (product of extremes)

6. Start the proportion by making x the second term of the second ratio.

$$: \; = \; : x$$

Fill in the first term of second ratio, remembering that both terms must be of the same kind. The answer represents barrels.

$$: \; = 15 : x$$

Fill in the terms of the first ratio, after deciding which is the larger term in the second ratio.

$$\$90 : \$30 = 15 : x$$

Then use **Rule 6.**

$$\$30 \times 15 = \$450$$
$$\$450 \div 90 = 5$$

Then 5 barrels is the answer.

Proof

$90 : 30 = 15 : 5$ (the proportion)
$30 \times 15 = 450$ (product of means)
$90 \times 5 = 450$ (product of extremes)

7. Start the proportion by putting x as the second term of the second ratio.

The x represents money that A will receive, so the first term of that ratio must be in the same terms. All of the quantities in this problem are about money, so we must reason out which amount is in the same ratio as the x. We can assume that if x represents what A will receive then the $756 (the amount of his claim) should be the first term. Thus

From the wording of the problem we can see that A will not receive the whole amount of his claim, so the 756 is larger than x. Then the first term of the first ratio will be the larger, and the proportion will be

$$\$7560 : \$3100 = \$756 : x$$

Use **Rule 6.**

$$\$3100\times\$756=\$2343600$$
$$\$2343600\div\$7560=\$310 \quad \text{(Ans.)}$$

8. In this problem the interest is an *effect* because the result (effect) of borrowing or loaning money is interest. The *causes* are the sums of money (on which the interest is figured) and the time (months). The required answer is a sum of money, so it is a cause. We start the proportion by putting x as the second term of the second ratio.

$$: \; = \; : x$$

If x is a cause then the first and second causes must be the first and second terms of the second ratio. Thus we would write the proportion in words as follows:

1st effect : 2nd effect = 1st cause : 2nd cause (x)

Now we can place the actual numbers in the proportion.

$$\$84 : \$21 = 30\times\$480 : 15\times x$$
or
$$84 : 21 = 14400 : 15\times x$$

We know that x represents the answer which is a sum of money. Therefore the first term of the second ratio must also be a sum of money. But we know that the causes are time (months) and "sums of money." Thus the first term of the second ratio (1st cause) must be the product of months and money, which is $30\times\$480$. The second term of the second ratio is also a product of months and money because it is a cause. Thus $15\times x$ is the second term.

By studying the problem we can see that the sum which gains \$21 in 15 months is bound to be smaller than the sum which gains \$84 in 30 months. Therefore the answer will be less than \$480. This means that the first term in the *second* ratio is larger than the second term. For this reason the first term of the *first* ratio must be larger than the second term. (We have already learned that if the first term of the first ratio is larger than the second term, then the first term of the second ratio must be larger than the second term.)

The two effects are \$84 and \$21. With the above explanation in mind, it is easy to see that \$84 must be the first term and \$21 the second term of the first ratio.

Now that the proportion is complete, **Rule 6** can be used to find x.

$$\frac{\cancel{\overset{\overset{1}{\cancel{3}}}{21}}\times\overset{3600}{\cancel{14400}}}{\underset{\cancel{12}}{\cancel{84}}}=3600$$

If $15\times x$ is 3600, then x is $3600\div15=240$.
The answer is \$240.

9. In this problem "men" and "days" are the causes, and "acres" are the effects. So we would write the proportion, in words, like this

1st effect : 2nd effect = 1st cause : 2nd cause (x)

The x represents the answer which must be in terms of men.

$$52.2 : 417.6 = 6 \times 5 : 12 \times x$$
or
$$52.2 : 417.6 = 30 : 12 \times x$$

The causes are products of days and men. The second term in the second ratio is $12 \times x$ because there are 12 days involved. The first term of the second ratio must also be about days and men (cause) because both terms of a ratio must be alike. Thus the first term (1st cause) in the second ratio is 6×5.

It is easy to reason that more men will be required to reap 417.6 acres than for 52.2 acres. Therefore the second term of the second ratio is the larger. Then the terms in the first ratio (effects) must be placed so that the larger quantity is the second term.

Now that the proportion is established, **Rule 6** can be used.

$$417.6 \times 30 = 12528 \text{ (product of means)}$$
$$12528 \div 52.2 = 240$$

Note: Remember the rules for dividing decimals here. Add one zero to the dividend because the divisor has a single figure following the decimal point.

If $12 \times x$ is 240, then x is $240 \div 12 = 20$.

The answer is 20 men.

10. Dividing the first term also divides the ratio. For example take the ratio $16 : 2 = 8$. If we divide 16 by 2 we get 8. Then the ratio would be $8 : 2 = 4$. The 4 shows that the ratio has also been divided.

PRACTICAL MATHEMATICS

Section 10

EQUATIONS—FORMULAS

Lesson 1

For Step 1, keep in mind the contents of the Introduction. For Step 2, learn the meaning and purpose of formulas and equations and the rules governing the solution of equations. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

An **equation** is a statement of equality between two quantities. You have already learned that when the sign of equality (=) is used, the quantities on one side of that sign equal or balance the quantities on the other side. Therefore, when the equality sign is used, you have an equation.

Expressed in other words this simply means that an **equation** is a means of showing that two numbers or two groups of numbers are equal to the same amount.

In proportion principles, which you have already studied, you learned that the ratio on one side of the equality sign must equal the ratio on the other side. For instance, in the proportion 3:6::4:8 the ratio $\frac{3}{6}$ is equal to the ratio $\frac{4}{8}$ because $\frac{3}{6}$ when reduced to lowest terms equals $\frac{1}{2}$ and $\frac{4}{8}$ reduced to lowest terms equals $\frac{1}{2}$.

This can be expressed or written $\frac{3}{6}=\frac{4}{8}$. This is said, therefore, to **balance.** At this point imagine that Fig. 1 is a rough drawing of a scale such as is used to weigh groceries.

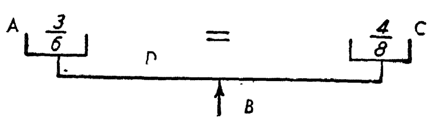


Fig. 1

The weighing pans are A and C. The balance arm is D. At B is the balance point. In order for the arm D to be perfectly horizontal, the weights in both pans, A and C, must be equal. This simple balance principle can be used to illustrate equations.

In pan A we have $\frac{3}{6}$ and in pan C we have $\frac{4}{8}$. Because both of these fractions are equal (both being equal to $\frac{1}{2}$) we can think of them as being **balanced.** This doesn't mean that they weigh the same, but it does mean that they are **equal.** So we can now state the rule that all **equations must balance.**

With this in mind, it is easily seen that $\frac{3}{6}=\frac{4}{8}$ is an equation. Both fractions are equal to the same thing, or, in other words they balance. Other simple equations are illustrated in the multiplication tables. Thus, $8\times7=56$ is an equation, because 8×7 is 56. Or, if we put 8×7 in the A pan and 56 in the C pan, in Fig. 1, the scale would balance, because 8×7 is exactly the same as 56. Other forms of simple equations are as follows: $6+5=11$; $12-5=7$; $20\div5=4$; $.6\times.4=.24$; $\sqrt{4+5}=1+\sqrt{4}$; $3^2-2=2^3-1$. Thus, we see that equations may have various numbers on either side of the equal sign, connected by $+$, $-$, $\times$, $\div$, root, or power signs. But whatever combination of numbers are shown on either side of the equal sign, they must **balance** in order to become an equation. The student should make sure he understands the explanation of equations up to this point before going ahead.

A **formula** is an equation which contains or means a rule or principle. In electrical work we know that current equals electromotive force divided by resistance. This is really a rule or principle because it always holds true. We can write this rule in equation form as follows:

$$I=\frac{E}{R}$$

I means current. E means electromotive force. R means resistance. This is an equation because it balances. In Fig. 2, the letter I can be assumed as being in pan A and $\frac{E}{R}$ in pan C. This is also called a **formula.**

The only difference between an equation and a formula is that

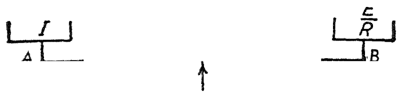


Fig. 2

in an equation the numbers generally haven't any particular meaning, whereas if an equation is composed of letters that mean something such as $I=\frac{E}{R}$ then it is called a formula. In other words a formula is a rule. In formula $I=\frac{E}{R}$, we know this is a rule for finding the current when we already know the electromotive force and resistance.

$7+2=4+5$ is an equation because it balances. The numbers have no particular meaning. $I=\frac{E}{R}$ is a formula because the letters all mean definite things and because it is a rule for finding the current.

As far as balance is concerned, or as far as method of working is concerned, equations and formulas are the same.

Formulas are the same as rules. We can say—**The current is equal to the electromotive force divided by the resistance.** But, this is a long and time taking rule to write each time we want to show it, so in all types of engineering work where we have given such names as **Current, Electromotive Force,** and **Resistance,** we substitute letters for them so, whenever we see these letters, we know what they mean. This allows us to write rules, as above, in short and easily written formulas.

Now, when we have formulas written in a simple form we can substitute actual numerical values for them and "solve the problem." The function of these lessons is to teach you how to substitute actual values in formulas and to solve such problems.

As a further illustration of balance, study the following material in Fig. 3. Here we have shown six different examples by first stating the form in words and then showing it by figures and also showing the balance by means of sketches.

Two added to three is equal to five
In formula form this is written $2+3=5$

From five take three and the result is equal to two
In formula form this is written $5-3=2$

Multiply two by three and the result is equal to six
In formula form this is written $2\times 3=6$

Six divided by three is equal to two
In formula form this is written $\frac{6}{3}=2$

Two raised to the second power is equal to four
In formula form this is written $2^2=4$

The square root of four is equal to two
In formula form this is written $\sqrt{4}=2$

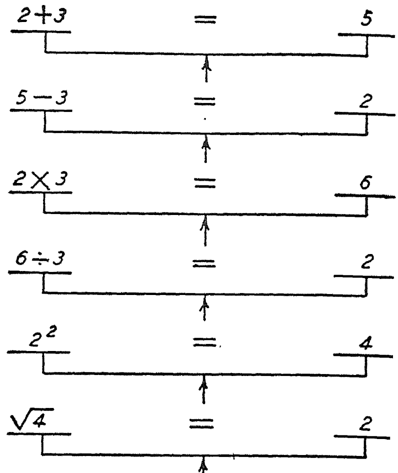


Fig. 3

When we use the formula $I=\frac{E}{R}$, we must substitute actual values in place of the letters before we can work it out in order to find one or another of the parts such as I, E, or R. Whenever we have such an actual problem to solve, where we have to substitute real values for these letters, it is a simple matter to tell what I, E, or R means as will be explained later. When we have substituted real values in the formula, the equation must balance or it is not correct.

Suppose we know that $I=20$, $E=120$, and $R=6$. Then the formula $I=\frac{E}{R}$ becomes $20=\frac{120}{6}$ because we have substituted the numbers in place of letters, like this

$$\overset{20}{\cancel{I}}=\frac{\overset{120}{\cancel{E}}}{\cancel{R}}$$

Here we first wrote down the formula and then crossed out the I and put the 20 in its place. Then we crossed out the E and put the 120 in its place. Also we crossed out the R and put 6 in its place. Such a procedure is called **substituting.** In this case, where we

know the value, in numbers, of all the letters in the formula we quickly see that the resulting equation balances because

$$\frac{120}{6}=120 \div 6=20.$$

So $20=20$, or the equation balances.

Now we come to the point where we can explain the real use of formulas and equations. (Remember that a formula is where only letters are given and an equation is where we have substituted actual number values for the letters.) In actual practice, when solving such formulas as $I=\frac{E}{R}$ we never know the value of more than two of the three letters. In other words there is always one letter the actual value of which we do not know. For example, we may know that the current is 40 and that the resistance is 12. But we do not know what the value of E is. The formula to start with is

$$I=\frac{E}{R}$$

Then we substitute the known values

$$\overset{40}{\cancel{I}}=\frac{E}{\underset{12}{\cancel{R}}}$$

This becomes—

$$40=\frac{E}{12}$$

The problem is then to find, by calculation, the value of E. Or, by another variation we may have

$$40=\frac{480}{R}$$

Here we must find the value of R by calculation.

At this point the student can begin to see how equations and formulas are used to a great extent. In the following text material, we will study the means or ways of solving equations and formulas.

Lesson 2

For Step 1, recall the principles involved in formulas and equations. For Step 2, learn the method of solving formulas and equations by placing the unknown by itself on one side of the sign of equality. For Step 3, study the Illustrative Examples. For Step 4, work the Practice Problems.

In $2+3=5$ we have an equation. (It is called an **equation** because it isn't stating any rule and because the numbers do not mean anything in particular.) In this equation we know every number and the equation balances. But, suppose we had $2+?=5$. We do not at first know the value of the "?." We say to ourselves, "What number must be added to 2 to get 5?" As the equation stands, it does not balance. We are to find the missing number which added to 2 will make the equation balance. In this simple illustration we can immediately see that the missing number is 3. Then $2+3=5$ and the equation balances. Now, what we really did was to find the missing number which subtracted from 5 gave 2. Finding this value is solving the equation.

$$(1)\quad 2+?=5$$

or

$$(2)\quad 2=5-?$$

$$(3)\quad 2=5-3$$

In the third equation we have shown that 3 is the required number. To do this, we have **changed the equation around** a little, as will be explained further.

You learned how to calculate the area of a square or rectangular field in the book on Denominate Numbers. If the area of a rectangle is 36 and one side is 9 and the unknown side is ?, then $?\times 9=36$, or $?=36\div 9=4$. Thus you see an unknown number can be found if enough of the other numbers are known. This feature was also discussed in the lesson on Proportion. The number missing is called the **unknown.** The part on the left side of the = sign in a formula is called the **first member** and the part on the right side of the = sign is called the **second member.** Also in some cases the part on the left side of the equal sign is called the "left side of the equation" and the part on the right side of the equal sign is called the "right side of the equation."

There are a few necessary rules to follow in solving equations and formulas. These will be given and illustrated after which we can go ahead with practical illustrations.

Rule (A). *The letters that occur in a formula or equation must be given such numerical values as will make both members numerically equal when these values are substituted in the formula.*

As an illustration of Rule (A)

Consider the equation $12+6=2\times9$
Simplified, this becomes $18=18$

Here it can be assumed that the values of the equation have been substituted, resulting in $12+6=2\times9$. Both members (that is first and second or left and right members) are equal because when simplified both members equal 18. To simplify, means to add the 12 and 6 and to multiply 2×9. Or, it can be assumed to mean performing what the signs indicate. The signs are $+$, $-$, $\times$, etc.

Rule (B). *The same quantity may be added to or subtracted from both members without unbalancing the equation.*

As an illustration of Rule (B)

(1) Add 10 to each side $18+10=18+10$
And the equation is still true or $28=28$

In other words the equation still balances.

(2) Subtract 10 from each side $18-10=18-10$
The final result is still true or $8=8$

In other words the equation still balances.

(1) We added 10 to each side of the simplified equation using Rule (B). It is easily seen that after this has been done the equation still balances.

(2) We subtracted 10 from 18 in both sides of the equation, Rule (B), and find after simplifying that $8=8$ so our equation still balances. It should be noted that in the above illustrative examples the amount changes such as $18=18$ and $8=8$, etc., but the main point is that no matter which of these examples we apply the equation *still balances.*

Rule (C). *Both members of a formula or equation may be multiplied or divided by the same quantity without destroying the balance.*

As an illustration of Rule (C)

(1) Multiply both sides by 7	$18 \times 7 = 18 \times 7$
Again the result is still true	or $126 = 126$
(2) Divide each side by 9	$18 \div 9 = 18 \div 9$
The result is still equal	or $2 = 2$

(1) We multiplied both sides of the simplified equation, Rule (C), by 7 and the equation still balances.
(2) We divided both sides of the same equation, Rule (C), by 9 and it still balances.

Rule (D). *Both members of a formula or equation may be raised to the same power without destroying the balance.*

As an illustration of Rule (D)

Suppose we again take the simplified equation given under Rule (A). We have $18 = 18$. Now raise both to the third power. $18^3 = 18^3$ or $5832 = 5832$. Also $18^4 = 18^4$, $18^7 = 18^7$, etc. The equation will still balance.

Rule (E). *The same root may be extracted in both members without destroying the equality.*

As an illustration of Rule (E)

We will take the same simplified equation $18 = 18$. If we take the square root of each side we have $\sqrt{18} = \sqrt{18}$ or $4.2426 = 4.2426$ or we could say $\sqrt[3]{18} = \sqrt[3]{18}$ or $\sqrt[63]{18} = \sqrt[63]{18}$. In all cases the equation balances.

Rule (F). *The order of the numbers or the order of the letters is not important if the proper sign is given to each.*

As an illustration of Rule (F)

We may change the location of the letters or numbers on the same side of the equation if we carry their signs with them.

(1) For example:	$(6 \times 8) - 6 - 20 = 22$
(2) Changing locations	$-6 + (6 \times 8) - 20 = 22$
(3) or	$-20 + (6 \times 8) - 6 = 22$

At (1) we have an equation that balances because $6 \times 8 = 48 - 6 = 42$ and $-20 = 22$. Thus $22 = 22$. The parentheses ( ) around the 6×8 mean that 6×8 must be multiplied before subtracting the -6. At (2) the second member or right side of the equation remains un-

changed. But the first member or left side of the equation has been changed insofar as the -6 being moved so it comes before the (6×8) instead of after it. The results are still correct because $-6+(6\times8)=-6+48=42$. And $42-20=22$. In (3) the -20 has been moved but the equation still balances.

Remember that we can change the location of letters or numbers on the same side of the equation without changing signs. But if we move a number from the left side of the equation to the right side, or vice versa, it becomes a different matter as will be explained by the following rule.

Rule (G). *Numbers and letters can be moved or shifted from one member or side of an equation to the other member or side if their signs are changed from + to − or from − to +.* This is called **transposition.**
As an illustration of Rule (G)

The process of transposition is constantly used in the solution of formulas and equations. When there is no sign in front of a number or letter, + is understood. Thus, 8 means +8; 12 means +12.

ILLUSTRATIVE EXAMPLES

1. Transpose the first number in the left member of the equation, $7+5-3=6+3$, to the right member.
Here we have

$$7+5-3=6+3$$

Transposing we get

$$+5-3=6+3-7$$

or

$$+2=+2$$

The arrow shows how the transposition was done. Rule (G) says that the sign must be changed from + to minus when a number is shifted from one side of the equation to the other. The 7 without any sign means +7. Following the rule we changed the sign to (−) when we shifted the 7. Another rule meaning the same as Rule (G) is: *Whenever a number is moved or shifted from one side of the equation to the other, its sign must be changed to the opposite of what it was before the move or shift was made.*

2. Transpose the second and third numbers in the right member of the equation, $2+2=3+2-1$, to the left member.

Here we have

$$2+2=3+2-1$$

Transposing we get

$$2+2-2+1=3$$

or

$$3=3$$

The arrows show how the shift or transposing was done. The signs were changed to the opposite of what they were originally.

In both examples 1 and 2 the results balance perfectly.

Rule (H). *When transposing or shifting numbers from one side of the equation to the other side, the following two methods must be remembered.*

Method (1).

Any number or numbers being shifted or transposed from the first member or left side of the equation to the second member or right side of the equation must be annexed or written **after** what was originally on the right side of the equation. Thus in example 1, page 9, the 7 goes from the left side of the equation to the right side of the equation and is annexed or written after the 6+3.

Method (2).

Any number or numbers being shifted or transposed from the second member or right side of the equation to the first member or left side of the equation must be annexed or written **after** what is already in the first or left member of the equation. Thus, in example 2, page 10, the 2 and 1 are written after the 2+2.

To Sum Up

We can change or shift the numbers around on one side of an equation, or the other side, as much as we please just so long as we make sure their signs are not changed.

But, when we shift or transpose the numbers from one side of an equation to the opposite side, Methods (1) and (2) must be remembered and the plus or minus signs must be changed to the opposite of what they were before the shift or move was made.

These rules are not difficult if the student will remember them and refer to them often. The following practice problems should

now be worked out, but not until the student thoroughly understands all of Lesson 2 perfectly. This may require lots of time and hard study. Be sure to work *all* problems.

PRACTICE PROBLEMS

Work the problems without looking at the answers.

Transpose the last number in the right member of the equation to the left of the = sign:

1. $2+4=5+1$ Ans. $2+4-1=5$
2. $3+5=9-1$ Ans. $3+5+1=9$
3. $7-2=8-3$ Ans. $7-2+3=8$
4. $6-5=0+1$ Ans. $6-5-1=0$

Transpose the first number in the left member to the right member:

5. $6-5=1-0$ Ans. $-5=1-0-6$
6. $8+10=12+6$ Ans. $+10=12+6-8$
7. $5-9+1=-3$ Ans. $-9+1=-3-5$

Lesson 3

For Step 1, keep in mind the meaning of the words "unknown," "term," "expression," and their relation to other numbers with which they are found. For Step 2, learn how letters are used to indicate certain values in formulas and how to solve a simple formula. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Having learned, in Lessons 1 and 2, what equations and formulas are and the rules for shifting letters or numbers around in equations, and how to transpose numbers, we can now go ahead another step and learn how to solve an equation when all but one of the quantities or parts are known. In Problems 1 and 2, pages 9 and 10, all quantities were known. In other words, there were no missing or unknown parts. In actual practice where formulas and equations are used in engineering work, for example, the equations are formed from standard formulas and in every case there is one part unknown as indicated on page 5. Thus, we see the real benefit of equations and of transposing. The basic use of equations is to find missing parts or unknown numbers, as will be shown later.

When all but one of the quantities of a formula or equation are given, we can often make use of the principle of transposition

to solve the equation. In the equation $2+?=5$, we found that the value of "?" is 3, through a simple process of reasoning; but when the formulas are more complicated, we can make use of the principle of transposition.

ILLUSTRATIVE EXAMPLES

1. $?+3\frac{12}{27}=6$

Here we have the problem of finding the value of the "?."

Rule (I). *To find the missing number or to balance the equation proceed as follows: If the unknown number (represented by "?") is on the left side of the equation (or on the left side of = sign), let it remain there but move all other known numbers to the right side of the equation. If the missing number is on the right side of the equation, move it to the left side and move all known numbers so they will all be on the right side of the equation.* (Remember Rules (F) and (G) and Methods (1) and (2) following Rule (H)).

Now, to solve Problem 1 we write it out as given

$$?+3\frac{12}{27}=6$$

$$?=6-3\frac{12}{27}$$

Then following first part of Rule (I) we leave the (?) where it is and move the $3\frac{12}{27}$ to the right side of the equation.

The arrow shows how the $3\frac{12}{27}$ was moved from the left to the right side of the equation. We changed the sign from + to − applying Rule (G). The $-3\frac{12}{27}$ was put after the 6 applying Method (1) following Rule (H).

Proof. Now if we actually subtract $3\frac{12}{27}$ we get $2\frac{15}{27}$. Thus the "?" becomes $2\frac{15}{27}$. Then

$$2\frac{15}{27} = 6 - 3\frac{12}{27}$$

and our equation balances and we have solved the problem. Now go back over Problem 1 again to make sure you understand it.

2. $6 = ? + 2\frac{15}{27}$

Here we have another equation in which we have a missing number which is represented by "?." To work Problem 2, we use the last part of Rule (I). Thus the "?" is moved to the left side of the equation and the 6 is moved to the right side.

We write the problem as given

$$6=?+2\frac{15}{27}$$

Then we follow Rule (I)

$$-?=2\frac{15}{27}-6$$

The "?" becomes $-?$ applying Rule (G). The 6 becomes -6 also applying Rule (G). The -6 is placed after the $2\frac{15}{27}$ applying Method (1) following Rule (H). Solving by arithmetic we find that $2\frac{15}{27}-6=-3\frac{12}{27}$. Then $-?=-3\frac{12}{27}$ or $?=3\frac{12}{27}$. Thus the $3\frac{12}{27}$ is the quantity or missing part we wanted. It might be explained that $2\frac{15}{27}-6$ is the same as $-6+2\frac{15}{27}$.

Rule (J). *In cases like the above problem where $-?=-3\frac{12}{27}$, we have a minus sign for both sides so in the final answer it becomes plus.*

We can also work Problem 2 in another way as follows:
Using our sign principles, we change the $2\frac{15}{27}$ from the right to the left of the equal sign and leave the ? all by itself. Thus, $6-2\frac{15}{27}=?$
Solving this left side by Arithmetic, we get $3\frac{12}{27}=?$ or, in words, $3\frac{12}{27}$ is the quantity we needed in place of the question mark in order to have all the facts or to balance the equation.
Here, to balance the equation, the number missing was on the right side of the equal sign, and we changed all the known numbers that were on the right to the left of the equal sign and thus left the question mark all by itself on the right side. We solved the left side by Arithmetic, and the answer was the value of the question mark.

Either way of solving Problem 2 is correct. The first method of solution uses the second part of Rule (I) whereas the second method of solution doesn't use Rule (I). The student is advised to become familiar with both methods of solution for problems such as Problem 2 where the unknown or missing quantity is on the right side of the equation to start with. When the missing quantity is on the left side of the equation to start with, as in Problem 1, then always use the method of solution given for that problem.

You have already noticed that when a number is all by itself and it is +, the sign is omitted. Also, when a number is the first of a group of numbers on one side of the equal sign, and it is +, the

sign is omitted. This is just a convention for simplicity. The + sign is the only one that may be omitted, the − sign must always be written, and the + sign may be omitted only when, as already stated, the number is all by itself or is the first of the group.

$$+3-2=+1 \text{ is the same as } 3-2=1$$
$$-3+4=+1 \text{ is the same as } -3+4=1$$

PRACTICE PROBLEMS

Find the value of ? in each equation by the method of leaving the ? by itself. In Problems 1, 2, 4, and 5 use both methods of solution as explained for Problem 2, page 12.

1. $5+3+1=8+?$ Ans. $?=1$
2. $6+1+5=10+?$ Ans. $?=2$
3. $?+7-2=8+0$ Ans. $?=3$
4. $7-2+3=?+8$ Ans. $?=0$
5. $2\frac{1}{2}+?=7$ Ans. $?=4\frac{1}{2}$
6. $3\frac{1}{15}+?=4+1$ Ans. $?=1\frac{14}{15}$
7. $1\frac{1}{3}-\frac{2}{3}+?=1+7-3-4$ Ans. $?=\frac{1}{3}$

Lesson 4

For Step 1, recall the method of substituting given values for letters in a simple formula. For Step 2, learn how to indicate the multiplication of two or more letters, and how this principle is used in formulas. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

So far, we have used the question mark (?) for the missing number. Instead of the question mark, we could have used any other symbol or mark. The question mark was an arbitrary symbol selected by us. The symbol generally used is one of the letters of the alphabet. Suppose we select the letter x. Then, we should write the equation $5+3+1=8+?$ in this form: $5+3+1=8+x$. In the same way in the other equations, we could have substituted an x for the question mark.

Also, we have used the words "number" and "figure" to designate the quantities separated by + and − signs. The proper name is **term** and we shall use it from now on. Thus in the equation $5-3+2=8-x$, each one of the numbers, 5, 3, 2, 8, and the letter x, is called a term. A term must always have a + or − sign before

it either expressed or understood. (We have mentioned the cases where $+$ may be omitted.) A group of related terms is usually called an **expression.** If a number is enclosed with other members within a parentheses, ( ), the part within the parentheses must be considered a single quantity or unit. The various terms cannot be dealt with separately. Thus, in the expression $3+(2+5-7)$, the part $(2+5-7)$ is a unit; and in $6-(3-8)$, the part $(3-8)$ is a unit. Similarly if several terms form the numerator or denominator of a fraction, each expression must be considered a single quantity. For example, in $4+\frac{5-3}{2+5}$, the 5 and the 3 of the numerator are inseparable, as are also 2 and 5 in the denominator. They are as much a unit as are 4 and 7 and 6 and 4 in the fraction $\frac{47}{64}$. You could not deal with the 4 apart from the 7 nor with the 6 apart from the 4.

So far we have used numbers only in our equations, with the exception of x, the term to be found. We could just as well have used the names of the figures instead. For instance, we could have stated $5+3+1=8+x$ as follows: Five $+$ three $+$ one $=$ eight $+$ x. Still further, we could have used the first letter of each one of the words instead of the word itself, thus, $F+T+O=E+x$, provided that we always kept in mind throughout the operations that each letter stood for the name of the number used, and naturally for the number itself.

This is just exactly what we do in writing formulas. We use letters to indicate names or quantities and connect them by the signs of operation, just as we did with our numerical practice formulas.

To solve these formulas with letters, we first use the numerical values of all the letters possible. We substitute these numerical values for the letters in the formula and then solve for the letter that remains, and for which we have no numerical value, in the same way that we solved for "?" in our practice formulas.

Note: The student should understand that formulas can be solved, even though the meaning of the letters is not known. So long as we know the values of a sufficient number of the letters, we can solve the formula. The meaning of the letters is given in the formulas that are discussed in this book, but that information is not necessary to the solution of the formula itself. It is necessary, however, in order to solve a written problem involving a formula.

In the many different branches of engineering, for example, we make use of formulas and equations very frequently. By actual tests, engineers have found that one formula will hold good for any condition. Just to illustrate this, we will use a formula connected with electrical engineering. It is not necessary that the student understand the principles of electricity to work with and understand these mathematical formulas. We will take the formula $I=\frac{E}{R}$. We used this same formula on pages 2 and 5 in explaining what formulas were.

In the above formula, the relationship is always exactly the same. This was proven by actual tests many years ago. So, in electrical work, if we know the exact values of E and R, we can always find I. If we know the values of I and R, we can find E, and if we know the values of I and E, we can always find R. Thus, it can be seen that formulas save much time because without them we would have to make actual tests, at great expense of time and money, to find I, E, or R depending on which one was missing or unknown. We will try a few illustrations to see how the formulas work.

ILLUSTRATIVE EXAMPLES

1. If $E=50$ and $R=5$, what is the value of I?

Note: In electrical work this is a typical problem. It often happens that we know the value of E and R and desire to find the value of I.

To solve Problem 1, we take formula

$$I=\frac{E}{R}$$

The first step in solving Problem 1 is to substitute the known values for the letters of known value.

$$I=\frac{\overset{50}{\cancel{E}}}{\underset{5}{\cancel{R}}}$$

In place of the E we put 50 and in place of the R we put 5. Then we have

$$I=\frac{50}{5}$$

The next step is to perform all indicated operations. The 50 over 5 means 50 divided by 5. Whenever you see one number over another number, it indicates that the top number is to be divided by the lower number.

$$50 \div 5 = 10$$

2. If $R=5$ and $I=5$, what is the value of E?

The solution of Problem 2 is a little more difficult, but it requires only the things you have learned in Lessons 1, 2, and 3. The formula is

$$I = \frac{E}{R}$$

Substitute the known values for R and I in the formula

$$\overset{5}{\cancel{I}} = \frac{E}{\cancel{R}} \quad \text{or} \quad 5 = \frac{E}{5}$$

Now, we want to find the value of E. To do this we perform what is called "clearing of fractions." The $\frac{E}{5}$ is the fraction. Turn back to Rule (C) on page 7 and review it carefully. This rule says we can multiply both sides or members of an equation by the same quantity. In cases like this problem it is best to use as that quantity a number equal to the denominator of the fraction. So, we will multiply both sides of the equation by 5, and get

$$5 \times 5 = \frac{E}{5} \times 5$$

Now we can cancel

$$5 \times 5 = \frac{E}{\cancel{5}} \times \cancel{5}$$

because the upper 5 is the same as saying $_1$ (If cancellation is not clear refer to Section 2.)

Now we have

$$5 \times 5 = E \quad \text{or} \quad 25 = E$$

3. If $I=1$ and $E=32$, what is the value of R?
The formula is

$$I=\frac{E}{R}$$

Substituting

$$\overset{1}{\cancel{I}}=\frac{\overset{32}{\cancel{E}}}{R} \quad \text{or} \quad 1=\frac{32}{R}$$

Refer back to Rule (C) again. We can divide both sides of the equation by the same quantity without destroying the balance. Thus

$$\frac{1}{32}=\frac{32}{R\times 32}$$

$\frac{1}{32}$ is the same as $1\div 32$. We put it in the form of a fraction so that we can cancel. When we divide $\frac{32}{R}$ by 32, we write it as shown because R is already the denominator of $\frac{32}{R}$ and not knowing the value of R we can't do anything more than indicate that R must be multiplied by 32. We use 32 as a divider because there is already a 32 in the equation and by using another one we can make cancellation possible.

Cancelling—

$$\frac{1}{32}=\frac{\overset{1}{\cancel{32}}}{R\times \underset{1}{\cancel{32}}}$$

So

$$\frac{1}{32}=\frac{1}{R}$$

Now, if $\frac{1}{32}=\frac{1}{R}$ then 32 must $=R$. Ans. is 32 or $R=32$.

In the formula $I=\frac{E}{R}$, there are three letters and any one of

the letters may be unknown. In Problems 1, 2, and 3 the method for solving for all three has been given and explained. No matter what the values of the letters in this formula are the method of solving is exactly the same as explained for the three problems. Therefore the student should memorize these three solutions the same as a rule.

PRACTICE PROBLEMS

1. If $E=80$ and $R=40$, what is I? Ans. I=2
2. If $R=15$ and $I=15$, what is E? Ans. E=225
3. If $I=3$ and $E=96$, what is R? Ans. R=32

ILLUSTRATIVE EXAMPLES

When you studied percentage, you learned three rules—one to find the percentage, one to find the base, and one to find the rate. Now that you know how to solve formulas, you may use only one of the rules and get all the parts from it. The formula is this:

$$\text{Percentage} = \frac{\text{Rate} \times \text{Base}}{100} \qquad \text{or in symbols: } P=\frac{R\times B}{100}$$

1. Find P, when $R=15$ and $B=5000$

The formula is

$$P=\frac{R\times B}{100}$$

Note: The figure 100 is called a **constant.** No matter whether we are solving to find P, R, or B this constant remains in the same position. In Percentage you learned that per cents were in terms of 100. That is why the 100 is used.

To go on with the problem, the first step is substitution. Substituting the given values for R and B

$$P=\frac{\overset{15}{\cancel{R}}\times\overset{5000}{\cancel{B}}}{100}$$

or

$$P=\frac{15\times 5000}{100}$$

Next we can cancel

$$P=\frac{15\times \overset{50}{\cancel{5000}}}{\cancel{100}}$$

Then $P=15\times 50$ or 750.

2. When P is 300, B is 15000, what is R?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Write the formula	$P=\frac{R\times B}{100}$
Substitute the given values for P and B in the formula	$300=\frac{R\times 15000}{100}$
Step 2	**Step 2**
To free the equation from fractions, multiply both members by 100, since 100 is the denominator of the fraction. Rule (C).	$300\times 100=\frac{R\times 15000\times 100}{100}$
Step 3	**Step 3**
Perform the cancellation	$300\times 100=\frac{R\times 15000\times \cancel{100}}{\cancel{100}}$ $30000=R\times 15000$
Step 4	**Step 4**
If 15000 times R is 30000, $R=30000\div 15000$ R is 2% Ans.	$30000\div 15000=2$

3. Given: $P=250$ and $R=10$. Find the value of B.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Formula is	$P=\frac{R\times B}{100}$

Substitute the given values for P and R in the formula

$$250=\frac{10\times B}{100}$$

Step 2

Multiply both members of the equation by 100 to clear of fraction. Rule (C).

Step 2

$$250\times 100=\frac{10\times B\times 100}{100}$$

Step 3

Perform the cancellation

Step 3

$$250\times 100=\frac{10\times B\times \cancel{100}}{\cancel{100}}$$

$$25000=10\times B$$

Step 4

If 10 times B is 25000, $B=25000\div 10$

$B=2500$ Ans.

Step 4

$$25000\div 10=2500$$

Note: $B\times 10$ means B taken 10 times (or 10 B's); $B\times 150$ means B taken 150 times (or 150 B's). We may omit the sign $\times$ between the figure and the letter and indicate the multiplication by placing the figure first and the letter next without any sign between. Then 10 times B would be written as $10B$; 150 times B as $150B$; 50 times X as $50X$, and so on.

PRACTICE PROBLEMS

1. Given formula $P=\frac{R\times B}{100}$. R is 9, B is 3000. Find P. 270 Ans.
2. Using the formula for Problem 1, find R when $P=30$ and $B=1000$. 3% Ans.

Note: The student may have already noticed that the letter R has appeared in two different formulas. This is perfectly all right because R has a different meaning in each formula. In electrical work R always means one thing and in percentage work R always means rate.

Since letters stand for values or numbers, we may indicate the multiplication of two or more letters by placing them beside each other without any sign between them. For example, take the formula for the horsepower of a steam engine which is $\text{HP}=\frac{PLAN}{33000}$. This means that the values of P, of L, of A, and of N are to be

multiplied together and then this result is to be divided by 33000, to get the amount of horsepower, or the value of HP. The 33000 is a **constant** and appears whenever this formula is used. It is part of the formula and was created at the time the formula was made by actual test.

ILLUSTRATIVE EXAMPLES

1. Find HP when $P=75$, $L=1.33$, $A=78.54$, and $N=180$.

Substituting these values in the formula $\text{HP}=\frac{PLAN}{33000}$, we have

$$\text{HP}=\frac{\overset{75}{\cancel{P}}\quad\overset{1.33}{\cancel{L}}\quad\overset{78.54}{\cancel{A}}\quad\overset{180}{\cancel{N}}}{33000}$$

or

$$\text{HP}=\frac{75\times 1.33\times 78.54\times 180}{33000}$$

All of these numbers must be multiplied starting at the left.
$75\times 1.33=99.75$
$99.75\times 78.54=7834.3650$
$7834.3650\times 180=1410185.7000$
This figure 1410185.7000 is then divided by 33000.

$$1410185.7000\div 33000=42.7329$$

Thus $\text{HP}=42.7329$

2. In the formula $\text{HP}=\frac{PLAN}{33000}$, $\text{HP}=80$, $P=75$, $A=160$, $N=180$. What is the value of L?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Write the formula	$\text{HP}=\frac{PLAN}{33000}$
Substitute the given values in the formula	$80=\frac{75\times L\times 160\times 180}{33000}$

Step 2

Multiply the known numbers in numerator together and divide by 33000

Step 2

$(75 \times 160 \times 180) \div 33000 = 65.45$

Then, $80 = 65.45 \times L$

Step 3

If 65.45 times L is 80, then L equals $80 \div 65.45$
$L = 1.22$ Ans.

Step 3

$80 \div 65.45 = 1.22$

3. What is the value of A, if HP=80, $P=75$, $L=1$, $N=180$, in the formula $\text{HP} = \dfrac{PLAN}{33000}$?

Substitute the given values

$$80 = \frac{75 \times 1 \times A \times 180}{33000}$$

Step 1

Multiply the known numbers in the numerator together and divide by 33000

Step 1

$(75 \times 1 \times 180) \div 33000 = .409$
Then, $80 = .409 \times A$

Step 2

If .409 times A is 80, then A equals $80 \div .409$
$A = 196$ Ans.

Step 2

$80 \div 409 = 196-$

PRACTICE PROBLEMS

In the following problems, the formula is $\text{HP} = \dfrac{PLAN}{33000}$

1. Find HP when $P=78$, $L=1.5$, $A=113$, and $N=170$. Ans. 68.11
2. Find P when HP=80, $L=1.22$, $A=160$, and $N=180$. Ans. 75+
3. Given HP=100, $L=2$, $A=1$, and $N=150$. Find value of P. Ans. 11000

Lesson 5

For Step 1, bear in mind the meaning of the square of a number and review the extraction of square root. For Step 2, learn the meaning of the word "constant" as used in formulas; study the relation of the parts of a number containing the square of a letter. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

Neglecting the resistance of the air, the formula for finding the space traveled by a body falling from rest through space for a certain length of time is

$$S=\tfrac{1}{2}gt^2$$

In this formula the letter g is equal to a value that is always the same. It is another example of a constant. As explained, constants frequently occur in formulas, and their values are always given when a solution of the formula is required.

We are now going a step in advance in learning about formulas and equations and assign real values to the letters, and solve problems where we are required to reason out the proper values.

In the above formula, S is the space in feet, t is the time in seconds, and g is the constant and represents the acceleration due to gravity or the increase in speed due to a pull from the earth. The value of g is always 32.16 feet per second. That is, every succeeding second that the body falls, it falls 32.16 feet faster than in the previous second. The $\frac{1}{2}$ in the formula is for the purpose of reducing the amount of travel to an average figure. An object, such as a piece of iron, if dropped from the top of a building gradually gains speed. From the point where it starts, it is or has been at rest or motionless so at the end of the first second it will not have traveled 32.16 feet, but only one-half of that amount or the average. The student need not worry about this average in learning how to use the formula, but it has been briefly explained in case any questions should arise.

We notice that the $\frac{1}{2}$, the g, and the t^2 are side by side without any sign between them, therefore, we know that they are all multiplied together. The power sign in a formula affects only the letter or number or parentheses above which it is found, so the t, only, is squared. The formula might be written $S=\frac{1}{2}\times g\times t^2$. Therefore, to solve the formula for S, we square the value of t and then multiply the result by the other values.

ILLUSTRATIVE EXAMPLES

1. In the formula $S=\frac{1}{2}gt^2$, what is S if the value of t is 25 seconds?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Write the formula	$S=\frac{1}{2}\times g\times t^2$
	or
Substitute the given values in the formula	$S=\frac{1}{2}\times \overset{32.16}{\cancel{g}} \times \overset{(25)^2}{\cancel{t^2}}$
	or
	$S=\frac{1}{2}\times 32.16\times (25)^2$
	The 25 is placed in ( ) so no mistake will be made whereby some other numbers might be included in the squaring
Step 2	**Step 2**
Cancel, and square 25	$S=\frac{1}{\cancel{2}}\times \overset{16.08}{\cancel{32.16}}\times 625$
	$S=16.08\times 625$
Step 3	**Step 3**
Perform the multiplication $S=10050$ Ans.	$16.08\times 625=10050.00$

2. Using the same formula as in Problem 1, find t if $S=5280$ feet.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Write the formula	$S=\frac{1}{2}\times g\times t^2$
Substitute the given values in the formula	$5280=\frac{1}{2}\times 32.16\times t^2$
Step 2	**Step 2**
Perform the cancellation	$5280=\frac{1}{\cancel{2}}\times \overset{16.08}{\cancel{32.16}}\times t^2$
	$5280=16.08\times t^2$
Step 3	**Step 3**
If 16.08 times t^2 is 5280, t^2 will equal 5280 divided by 16.08	
Perform the division	$5280\div 16.08=328.35$
$t^2=328.35$	
Step 4	**Step 4**
If the square of t is 328.35, t will be the square root of 328.35. Rule (e). Find the square root of 328.35	$\sqrt{328.35}=18.1$
18.1 sec. Ans.	

3. A weight is let drop from an airplane at a height of 600 feet. In what time will the weight strike the ground?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Write the formula	$S=\frac{1}{2}gt^2$
In our problem, S, the space the weight falls, is 600 feet. g is always 32.16. Substitute these values in the formula	$600=\frac{1}{2}\times 32.16\times t^2$
Step 2	**Step 2**
Perform the cancellation	$600=\frac{1}{\cancel{2}}\times \overset{16.08}{\cancel{32.16}}\times t^2$ $600=16.08\times t^2$
Step 3	**Step 3**
If 16.08 times t^2 is 600, then t^2 will equal 600 divided by 16.08	
Perform the division $t^2=37.31+$	$600\div 16.08=37.31+$
Step 4	**Step 4**
Use rule (e). Take the square root of both sides of equation 6.1 sec. Ans.	$t=\sqrt{37.31}=6.1+$

PRACTICE PROBLEMS

1. In the formula $S=\frac{1}{2}gt^2$, find t when $S=25000$ feet.
Ans. 39.4 sec.

2. A stone is dropped from the top of a building. It reaches the pavement in 3.2 seconds. How high is the building?
Ans. 164.6+ ft.

3. How many seconds will it take for a weight to fall to the ground from a height of 50 feet? Ans. 1.7+ sec.

Lesson 6

For Step 1, review Lesson 3, to fix in mind the relation of a number of terms when found in the denominator of a fraction. For Step 2, study the form of a formula involving such a group of numbers. For Step 3, work the Illustrative Example. For Step 4, work the Practice Problems.

In the study of electricity many formulas are used. One of these was used in Lesson 3. Another is discussed here—the formula for multiple circuits in parallel.

When there are two resistances in parallel, the formula is

$$R=\frac{1}{\dfrac{1}{a}+\dfrac{1}{b}}$$

in which R represents the total resistance and a and b the individual resistances.

When there are three resistances in parallel, the formula is

$$R=\frac{1}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}$$

In general, for any number of resistances in parallel, the formula is

$$R=\frac{1}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}+\dfrac{1}{e}+\cdots}$$

using as many different letters as there are individual resistances. The dots are used to show that the letters are continued to as many as are desired. They do not affect any problem where the values of the letters are given.

It must be borne in mind, as taught in Lesson 3, that in this formula all the fractions in the denominator taken together form a unit and must be combined into one fraction before any other operations are performed. We cannot deal with these fractions apart from each other, because they represent a single quantity.

ILLUSTRATIVE EXAMPLE

1. In the formula $R = \dfrac{1}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}+\dfrac{1}{e}}$ find R when, a is 2, b is 3, c is 4, d is 5, and e is 6.

Solution

Instruction / *Operation*

Step 1

Substitute the given values in the formula

Step 1

$$R=\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}}$$

Step 2

Combine the fractions in the denominator

Step 2

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{87}{60}$$

$$R=\frac{1}{\frac{87}{60}}$$

Note: The $\frac{87}{60}$ is obtained after finding the L.C.D. for the fractions. We find 60 is the L.C.D. After expressing all the fractions in terms of this L.C.D., they are added and equal $\frac{87}{60}$. (For review, refer to Section 3.)

Step 3

$\dfrac{1}{\frac{87}{60}}$ means 1 divided by $\dfrac{87}{60}$

so perform the indicated division

$R=.689$ Ans.

Step 3

$$1\div\frac{87}{60}=1\times\frac{60}{87}=\frac{60}{87}=.689$$

Remember the rule for dividing fractions given in Section 4. Also to get .689 we divided 60 by 87. (For review, refer to Section 5.)

PRACTICE PROBLEMS

1. In the formula used in the Illustrative Example, find R when $a=3$, $b=5$, and $c=2$. Ans. .97−

2. Using the same formula, find R when a is 5, b is 3, c is 7, and d is 5. Ans. 1.14+

3. What is the joint resistance of four wires when connected in parallel if their separate resistances are 3, 5, 6, and 8 ohms, respectively? Ans. 1.2+

Lesson 7

For Step 1, bear in mind the meaning of a group of numbers enclosed in a parentheses. For Step 2, study the given formula that contains more than one kind of parentheses. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

The following formula leads us into the study of **parentheses.** We can see from it that many parentheses may be used inside of another parentheses. The outside parentheses act as parentheses of all the others inside of them. The parentheses inside must be worked first and finally the outside parentheses may be removed.

$$S=\frac{[(n-x)+(n-x)^2-(x-1)+(x-1)^2]P}{2n}+\frac{r(x-1)+(x-1)^2}{2n}.$$

Here we have a long formula wherein several letters are used. These letters represent mechanical factors which we are not concerned with as we are learning how to solve equations. In this formula we will substitute actual values just as we did in the previous lessons.

Note: Both large and small letters like E and e are used in formulas and have different meanings.

ILLUSTRATIVE EXAMPLES

1. Suppose in the above formula we know that

$$n=6$$
$$x=4$$
$$r=1000$$
$$P=2000$$
$$S=?$$

Then our problem is to find the value of S. We do this by substituting actual values in the formula and then solving as will be explained.

Step 1. Substituting.

$$S=\frac{\left[(\overset{6}{\cancel{n}}-\overset{4}{\cancel{x}})+(\overset{6}{\cancel{n}}-\overset{4}{\cancel{x}})^2-(\overset{4}{\cancel{x}}-1)+(\overset{4}{\cancel{x}}-1)^2\right]\overset{2000}{\cancel{P}}}{2\underset{6}{\cancel{n}}}+\frac{\overset{1000}{\cancel{r}}\;(\overset{4}{\cancel{x}}-1)+(\overset{4}{\cancel{x}}-1)^2}{2\underset{6}{\cancel{n}}}$$

Here we crossed off the letter n in each case and put a 6 in place of each n. We did this because $n=6$. Then we crossed off the letter x in each case and put a 4 in place of each x. Then we crossed off the P and put 2000 in its place and crossed off the r and put 1000 in its place. This is called *substituting*. We might use this same formula in solving a great many different problems, and in each different problem the values of n, x, r, etc. would all be different, but we would substitute the value just as shown under Step 1.

Now we can write the formula over again, but this time we will use the values of the letters just as we substituted them above.

$$S=\frac{[(6-4)+(6-4)^2-(4-1)+(4-1)^2]2000}{2\times6}+\frac{1000(4-1)+(4-1)^2}{2\times6}$$

A careful study of this will show that it is exactly the same as the formula on page 30 except that the letters have been replaced by their actual values. When you start to work out formulas in actual practice, the values of the letters will be given, or the problems stated in such a manner that the letter values won't be hard to find.

In the two denominators, which form part of this formula, we have 2×6. In the original formula (page 30) these denominators were $2n$. Then we substituted the 6 for the n in each case. We then put a "times" sign between the 2 and 6 or otherwise anyone would think it was 26 (twenty-six) instead of 2×6. We can put two letters or a number and letter side by side without a times sign between them when we want to indicate they are to be multiplied Thus—

nn means $n\times n$

$2n$ means $2\times n$

But when we substitute a figure such as 6, in place of the n (as in the second case) we must put a "times" sign between the 2 and 6.

Step 2. Perform operations inside of ( ) parentheses.

Taking the first set of parentheses, starting on left-hand end, we have $(6-4)$. Performing what is indicated we subtract 4 from 6 and have 2. Thus the $(6-4)$ becomes (2).

The next set of parentheses is $(6-4)^2$. Subtracting 4 from 6 we have 2. Then the $(6-4)^2$ becomes $(2)^2$. The square sign stays in its same position.

In like manner

$$(4-1) = (3)$$
$$(4-1)^2 = (3)^2$$

Now, the numerator of the first part of the equation (the formula becomes an equation after substitution takes place) is

$$[(2)+(2)^2-(3)+(3)^2]2000$$

In the second part of the equation, we form what is indicated within the parentheses and see that

$$(4-1) = (3)$$
$$(4-1)^2 = (3)^2$$

The numerator of the second part of the formula then becomes

$$1000(3)+(3)^2$$

Now we can write the equation

$$S=\frac{[(2)+(2)^2-(3)+(3)^2]2000}{12}+\frac{1000(3)+(3)}{12}$$

The number 12 in the two denominators is obtained by multiplying 2×6 in each case.

Step 3. The [ ] bracket or parentheses still holds the other parentheses inside, therefore, we must solve everything inside of the bracket before we can remove the bracket.

$$S=\frac{[2+4-3+9]2000}{12}+\frac{3000+9}{12}$$

Here we took away the ( ) and at the same time squared numbers where squaring was indicated. Also we multiplied the 1000 by the 3 in the second part of the equation.

Step 4. Combine the numbers inside the bracket

$$S=\frac{[12]\,2000}{12}+\frac{3009}{12}$$

The numbers $[2+4-3+9]$, shown in Step 3, equal 12, because $2+4=6-3=3+9=12$. The bracket is left on. Also $3000+9$, from Step 3 equals 3009.

Step 5. Perform the indicated operations

$$S=\frac{24000}{12}+\frac{3009}{12}$$

$$S=\frac{27009}{12}=2251 \quad \text{Ans.}$$

In Step 4 we had $[12]2000$.
Worked out this is $12\times2000=24000$.
Then adding 24000 and 3009 we get 27009.
Thus

$$S=\frac{27009}{12} \text{ or } 2251 \quad \text{Ans.}$$

In this manner we have solved the problem and found the value of S to be 2251.

Summary. The student is advised to study Problem 1 again very thoroughly. Then look through the following solution step by step to see if Problem 1 is clear.

Solution Problem 1

Formula

$$S=\frac{[(n-x)+(n-x)^2-(x-1)+(x-1)^2]P}{2n}+\frac{r(x-1)+(x-1)^2}{2n}$$

$$n=6 \quad x=4 \quad r=1000 \quad P=2000$$

Find S

$$S=\frac{[(6-4)+(6-4)^2-(4-1)+(4-1)^2]\,2000}{2\times 6}+\frac{1000(4-1)+(4-1)^2}{2\times 6}$$

$$S=\frac{[(2)+(2)^2-(3)+(3)^2]2000}{12}+\frac{1000(3)+(3)^2}{12}$$

$$S=\frac{[2+4-3+9]\,2000}{12}+\frac{3000+9}{12}$$

$$S=\frac{[12]\,2000}{12}+\frac{3009}{12}$$

$$S=\frac{24000}{12}+\frac{3009}{12}$$

$$S=\frac{27009}{12}=2251 \quad \text{Ans.}$$

The above solution for Problem 1, is the same as was given on page 30. But, we have left out all the explanations and shown only the main steps in the solution.

The student should study through this outline of the solution and make sure he understands each step thoroughly. If one is not sure of this outline as given, go back and review the explained solution on page 30.

When the student is sure he understands the problem, he should close the book and try to work the problem without looking in the book.

2. Using the above formula on page 30, find P when $n=8$, $x=5$, $r=1500$, and $S=5000$.

Step 1. Substitute the given values in the formula

$$5000=\frac{[(8-5)+(8-5)^2-(5-1)+(5-1)^2]P}{2\times 8}+\frac{1500(5-1)+(5-1)^2}{2\times 8}$$

At this point, substituting was carried on in the same manner as Step 1 in Problem 1 on page 31. The student is advised to do this work on scrap paper and make sure he understands Step 1 of this problem.

Step 2. Perform the operations inside the ( ) parentheses.

$$5000 = \frac{[(3)+(3)^2-(4)+(4)^2]P}{16} + \frac{1500(4)+(4)^2}{16}$$

This step is done exactly the same as Step 2 on page 32. For example, in this problem, the first set of ( ) is $(8-5)$. We know that $8-5=3$. So we have (3) as in Step 2. The 16's are obtained by multiplying 2×8.

Step 3. Solve the parts within the bracket

$$5000 = \frac{[3+9-4+16]P}{16} + \frac{6000+16}{16}$$

This comes from Step 2. We squared all the numbers where squaring was indicated. Then we multiplied 1500 by 4 to get 6000.

Step 4. Combine the parts inside the bracket

$$5000 = \frac{[24]P}{16} + \frac{6016}{16}$$

From Step 3 we have

$$[3+9-4+16]$$

This equals

$$3+9=12-4=8+16=[24]$$

The 6016 is obtained by adding 16 to 6000.

Step 5. Transpose, so as to get the term containing P by itself.

At this point the student should go back to page 9 and review through to page 34 and make sure that he thoroughly remembers transposition. In transposing, you will recall, we move certain terms of an equation around so as to get the unknown term on one side of the equation by itself. The term $\frac{[24]P}{16}$ in Step 4, being already to the right of the equal sign can be left in that position. The term 5000 is already on the left side of equation so it doesn't move. The term $\frac{6016}{16}$ is on the right side so when it is moved to the left side its sign must be changed. (Rule G)

Now we have

$$5000-\frac{6016}{16}\quad\frac{24P}{16}$$

The term $\frac{6016}{16}$ is the only one that was moved.

But the 5000 really means $\frac{5000}{1}$ when expressed as a fraction. We cannot subtract $\frac{6016}{16}$ from $\frac{5000}{1}$ because the two fractions do not have the same denominator. (In Section 3 you had addition and subtraction of fractions.) We could find the L.C.D. and change both fractions so that they would be expressed in terms of L.C.D. explained in Section 3. But we have an easier method. If we multiply both numerator and denominator of the $\frac{5000}{1}$ by 16, we get $\frac{16\times 5000}{16}$. This doesn't change its value a bit as can be proven by multiplying 16 by 5000 and dividing the product by 16.

Now our equation becomes

$$\frac{16\times 5000}{16}-\frac{6016}{16}=\frac{24P}{16}$$

Step 6. Combine the parts in left member of equation

$$\frac{80000}{16}-\frac{6016}{16}=\frac{24P}{16}$$

$$\frac{73984}{16}=\frac{24P}{16}$$

In this step the 80000 is obtained by multiplying 16 by 5000. The 73984 is obtained by subtracting $\frac{6016}{16}$ from $\frac{80000}{16}$. Thus $80{,}000-6016=73984$. (Recall or look up in Section 3 the rule for subtracting fractions.)

Step 7. Apply Rule (C). Multiply both members of the equation by 16.

In order to be able to finish our problem without complicated calculations, we can follow Rule (C). We multiply by 16 because

the equation already has the number 16 in it and we want to make cancellation possible. Thus we have

$$\frac{73984}{16}\times 16=\frac{24P}{16}\times 16$$

The number 16 can be written $\frac{16}{1}$. So we can cancel.

$$\frac{73984}{\cancel{16}}\times\frac{\cancel{16}}{1}=\frac{24P}{\cancel{16}}\times\frac{\cancel{16}}{1}$$

Then

$$73984=24P$$

and

$$P=73984\div 24$$
$$P=3082+.\quad \text{Ans.}$$

Summary. Once more the student is strongly advised to study the above detailed and explained solution again to make sure he understands it. Then look through the following solution step by step of Problem 2 to see if it is clear.

Solution Problem 2

Formula

Same as for Problem 1.

$$n=8\quad x=5\quad r=1500\quad S=5000\quad \text{Find } P$$

$$5000=\frac{[(8-5)+(8-5)^2-(5-1)+(5-1)^2]P}{2\times 8}+\frac{1500(5-1)+(5-1)^2}{2\times 8}$$

$$5000=\frac{[(3)+(3)^2-(4)+(4)^2]P}{16}+\frac{1500(4)+(4)^2}{16}$$

$$5000=\frac{[3+9-4+16]P}{16}+\frac{6000+16}{16}$$

$$5000=\frac{[24]P}{16}+\frac{6016}{16}$$

$$5000-\frac{6016}{16}=\frac{24P}{16}$$

$$\frac{16\times 5000}{16}-\frac{6016}{16}=\frac{24P}{16}$$

$$\frac{80000}{16}\quad\frac{6016}{16}-\frac{24P}{16}$$

$$\frac{73984}{16} \quad \frac{24P}{16}$$

$$\frac{73984}{16} \times 16 = \frac{24P}{16} \times 16$$

$$73984 = 24P$$

$$P = 73984 \div 24$$

$$P = 3082+. \quad \text{Ans.}$$

Study through the above main steps in solution to Problem 2 to make sure you understand all parts. If you do not understand all parts, review the detailed explanation starting on page 34.

As an example of how a student should check his knowledge, the following sample questions are given.

(a) Where do the two denominators of 16 come from?

(b) How is the [24]P obtained?

(c) Why are both sides of the equation multiplied by 16?

(d) Why is 16 used rather than 18?

The student should ask himself many such questions.

When the student feels that he fully understands the problem, he should try to solve it without the help of the book.

Note: In the formula we have been using in this lesson, the values of r, n, and x cannot be found without the use of Algebra.

PRACTICE PROBLEMS

Using the same formula as in the Illustrative Examples

1. Find S when $n=4$, $x=3$, $r=500$, and $P=2000$. Ans. 1125.5

2. If $S=3000$, $n=8$, $x=5$, and $r=1000$, find value of P. Ans. 1832.6+

Since there are great numbers of different formulas used in engineering and mechanical work of all kinds, it would be impossible to give instruction on the method of solving every possible formula. Many of them cannot be solved without a working knowledge of Algebra and Trigonometry.

However, if the student understands the principles discussed and illustrated in this book and will use good judgment and common sense, he will be able to solve many formulas of importance.

PRINCIPLES FOR READY USE

1. A formula is like a weight scale; the part on one side of the equal sign must have the same resultant value as the part on the other side. There must be a balance.

2. In a formula, any number or term may be transposed from the left to the right or from the right to the left of the equal sign if, in doing so, the sign of the number or term so transposed is changed to the opposite sign. The sign of a number or term is always in front of the number or term and is a part of it.

3. To find the unknown number, or to balance the formula, if the number belongs on the left side of the equal sign, change all the known numbers that are on the left to the right of the equal sign and thus leave the unknown number all by itself on the left side. Solve the right side by arithmetic, and the answer is the value of the unknown number. Reverse the procedure when the unknown number is on the right side.

4. Omit the + sign in front of a number when it is all by itself or when it is the first of a group of numbers on one side of the equal sign.

5. If a certain number has a + or a − sign in front of it but is enclosed in parentheses with other numbers or if it is above or below the dividing line in a fraction with other numbers, the operations indicated between the numbers must be performed before that number can be separated from the others.

6. If we add or subtract the same number from both members of an equation or formula or multiply or divide both members by the same number, the equality or balance of the formula does not change.

7. Multiplication between a number and a letter or between letters is indicated by placing them beside each other without any sign between them.

8. A number or letter beside a root sign when there is no sign between, indicates multiplication.

9. If we square the quantities on both sides of the equal sign or raise them to the same power, no matter what that power is, or if we extract the same root, the equality is not destroyed.

10. If two or more numbers, or two or more letters, or numbers and letters, or radical signs and letters, or radical signs and numbers, are to be multiplied together, and then the product raised to a power, each of the quantities to be multiplied together may be raised to the indicated power first and then the results multiplied.

11. In some formulas there appear letters used twice in the same formula such as T_1 and T_2. In such cases T_1 might mean temperature on the outside of a building and T_2 the temperature on the inside of the building.

12. A power sign in the formula affects only the letter or number or parentheses above which it is found.

13. A constant is a letter that always has the same known value.

14. When we have quantities of any kind anywhere connected by + and − signs, we may group them as one quantity by placing parentheses around all of them including their signs, and then dealing with the parentheses as if it were a single quantity or unit. Also, multiplication between quantities in parentheses is indicated by placing them beside each other without any sign between the parentheses.

15. Many parentheses may be used inside of other parentheses. The outside parentheses act as parentheses of all the other parentheses inside of it. The operations in the inside parentheses must be worked first and finally the outside parentheses may be removed.

EXAMINATION DIRECTIONS

Do not start to work the examination problems until you are absolutely sure you fully understand the entire text and until you have worked **all** of the practice problems.

When working examination problems, it is all right to refer to the text, but do not try to work these problems simply by comparing them with some of the illustrated problems. You should study the text until you thoroughly understand the principles then the examination will not be difficult.

All values, except unknowns, are given for all formulas in the examination problems.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on page 43.

If you miss more than two of the problems, it means you should review the whole Section carefully.

Do not try this trial examination until you have worked every practice problem in the Section.

Do not start the final examination until you have completed this trial examination.

1. In the formula $I = \dfrac{nE}{Re+nRi}$ find I when $n=20$, $E=1.08$, $Re=300$, and $Ri=4$.

2. In the formula $F=\dfrac{P_1P_2}{d^2}$, P_1 represents the number of units of magnetic strength in one pole, P_2 represents the number of units of magnetic strength in a second pole, d represents the distance between the two poles measured in centimeters; and F represents the force with which the two poles act upon each other.

(a) Find F when P_1 is 150, P_2 is 200, and d is 25.

(b) A pole of 50 units magnetic strength acts upon a second pole 10 centimeters away with a force of 30. What is the magnetic strength of the second pole?

3. In the formula $U = \dfrac{1}{\dfrac{1}{f_i}+\dfrac{x_1}{k_1}+\dfrac{x_2}{k_2}+\dfrac{x_3}{k_3}+\dfrac{1}{f_0}}$ find the value of U when $f_i=1.65$, $x_1=4$, $k_1=9.20$, $x_2=4$, $k_2=5$, $x_3=.5$, $k_3=3.3$, and $f_0=6$.

4. In the formula $U=\dfrac{Ur\times Uce}{Ur+\dfrac{Uce}{}}$ find the value of U when $Ur=.47$, $Uce=.38$, and $n=1.3$.

5. In the formula $r=\sqrt{\frac{1}{12}(a^2+b^2)}$ find the value of r when a equals 20 and b equals 70.

6. Find the value of x in the following equation.

$$3\tfrac{1}{5}-\tfrac{4}{5}-\tfrac{2}{5}+x=8-7-1+3$$

7. In the formula $G^p = \dfrac{W^E\times G^E+W^1\times G^1}{W^E+W^1}$ find G^p when $W^E=100$, $G^E=108.8$, $W^1=300$, and $G^1=77.9$.

8. In the formula $a=\sqrt{\dfrac{3LD}{4bF}}$ find value of a when $L=20{,}768$, $D=3.25$, $b=3$, and $F=7500$.

FINAL EXAMINATION

1. Write the following statements in formula form:

(**a**) The area of a circular ring is equal to 3.1416 multiplied by the result of subtracting the square of the radius of the inner circle from the square of the radius of the outer circle. Use R for radius of outer circle, r for radius of inner circle, A for area of ring.

(**b**) The volume of a cylinder is equal to the base area times the height. Use V for volume, a for base area, and h for height.

2. Solve the following for x:

(**a**) $3+5=5+x$
(**b**) $7+x-3=10-2$
(**c**) $9-3-2+4=x+0$
(**d**) $-6+1+3=2-x$

3. In the formula $P=\dfrac{H\times D+20}{100+E}$ find P when $H=5120$, $D=4$, and $E=100$.

4. In the formula $x=\dfrac{11+\overset{4}{\cancel{8}}Y}{\cancel{2}}$ can you cancel the 2 into the 8 as shown? Give a reason for your answer.

5. The load that a bolt can carry is found by the formula $W=\dfrac{S\pi d_1^2}{4}$ in which W is the load, S is the allowable stress per square inch, π is a constant (its value is 3.1416), and d_1 is the diameter at the root of the thread. Find W, when $S=7000$ and $d_1=.8$ inch.

6. A 120-volt lamp has a resistance of 250 ohms. What current will pass through the lamp?

7. The individual resistances in three wires connected in parallel are 3, 8, and 12 ohms, respectively. What is their joint resistance?

8. In the formula $H_t=AU(t-t_o)$ find H_t when $A=500$, $U=.25$, $t=70$ and $t_o=0$.

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. As explained in the text, we substitute real values for the letters in the formula. The problem gives us the formula

$$I=\frac{nE}{Re+nRi}$$

Then the values of the following letters are given: $n=20$, $E=1.08$, $Re=300$, $Ri=4$.

Before we can solve the formula to find the value of the unknown letter I, we must substitute real values. In the formula the term nE means n multiplied by E. We know the values of n and E, so we multiply $20\times1.08=21.60$. We know that $Re=300$. We know $Ri=4$, but this must be multiplied by n or 20. Thus $nRi=20\times4=80$. Thus in the above formula we can substitute

$$I=\frac{nE}{Re+nRi}\qquad\frac{21.60}{300+80}$$

Here we have just replaced the letters by their real values. Then

$$\frac{21.60}{300+80}\qquad\frac{21.60}{380}$$

Now we must divide 21.60 by 380.

```
380)21.6000(.0568
    19 00
     2 600
     2 280
       3200
       3040
        160
```

Thus the answer is .0568+

2. In the formula

$$F=\frac{P_1P_2}{\ }$$

(a) The letter values are: F = force or unknown, $P_1=150$, $P_2=200$, $d=25$. Substituting

$$F=\frac{150\times200}{25^2}$$

To make cancellation possible we write

$$F=\frac{\overset{6}{\cancel{150}}\times\overset{8}{\cancel{200}}}{\cancel{25}\times\cancel{25}}=48\text{ Ans.}$$

Next, multiply the fraction $\left(\frac{1}{12}\right)$ by the whole number (5,300).

$$\frac{1}{\cancel{12}}\times \overset{1325}{\cancel{5300}}=\frac{1325}{3}=441\tfrac{2}{3}$$

For ease in calculating we change the fraction to decimals, making 441.66. Now the equation is

$$r=\sqrt{441.66}$$

To find the value of r we must find the square root of 441.66.

$$\begin{array}{lllll}
 & 4\ 41.66\ 00\ 00 & \big|\, 21.015 \\
 & 4 & \\
41 & \quad 41 & \\
 & \quad 41 & \\
4201 & \qquad 66\ 00 & \\
 & \qquad 42\ 01 & \\
42025 & \qquad 23\ 99\ 00 & \\
 & \qquad 21\ 01\ 25 &
\end{array}$$

Thus $r=21.015$ Ans.

6. First write the equation

$$3\tfrac{1}{5}-\tfrac{4}{5}-\tfrac{2}{5}+x=8-7-1+3$$

Transpose

$$x=8-7-1+3-3\tfrac{1}{5}+\tfrac{4}{5}+\tfrac{2}{5}$$

The $3\frac{1}{5}$, $\frac{4}{5}$, and $\frac{2}{5}$ are moved to the right-hand side of the equation. In cases where we move figures from one side of the equation to the other, minus signs are changed to plus and plus signs to minus. Thus the $3\frac{1}{5}$ becomes $-3\frac{1}{5}$, the $-\frac{4}{5}$ becomes $+\frac{4}{5}$, and the $-\frac{2}{5}$ becomes $+\frac{2}{5}$. The x remains on the left side of the equation. Now we have to solve the equation in which the unknown value (x) is on one side and the known values are on the other side of the equation.

To solve the equation, add up all the plus numbers first.

$$8+3+\tfrac{4}{5}+\tfrac{2}{5}=11\tfrac{6}{5}=12\tfrac{1}{5}$$

Then add the minus numbers. These are added just as though they were plus.

$$-7-1-3\tfrac{1}{5}=-11\tfrac{1}{5}$$

Next subtract $-11\frac{1}{5}$ from $12\frac{1}{5}$.

$$\begin{array}{r}
+12\tfrac{1}{5} \\
\underline{-11\tfrac{1}{5}} \\
+\ 1
\end{array}$$

Thus

$$x=1 \quad \text{Ans.}$$

7. First write the formula

$$G^p = \frac{W^E \times G^E + W^1 \times G^1}{W^E + W^1}$$

Next, substitute the actual values

$$G^p = \frac{100 \times 108.8 + 300 \times 77.9}{100 + 300}$$

Next perform the indicated operations in the numerator of the equation. Always do what multiplying is necessary first. Then do the adding, thus,

$$100 \times 108.8 = 10{,}880$$
$$300 \times \ 77.9 = 23{,}370$$

Adding

$$10{,}880 + 23{,}370 = 34{,}250$$

Now our equation looks like this.

$$G^p = \frac{34250}{100 + 300}$$

Next perform the indicated operations in the denominator of the equation.

$$100 + 300 = 400$$

Now our equation looks like this

$$G^p = \frac{34250}{400}$$

Next divide 34,250 by 400

$$34250 \div 400 = 85.6 \quad \text{Ans.}$$

8. First write the formula

$$a = \sqrt{\frac{3LD}{4bF}}$$

Next substitute actual values in place of the numbers.

$$a = \sqrt{\frac{3 \times 20768 \times 3.25}{4 \times 3 \times 7500}}$$

Some cancellation is possible so as to simplify the figures in this equation (part

under square root symbol). To do this we will forget about the square root symbol for the time being.

$$\begin{array}{l}
\qquad 1298 \\
\qquad \cancel{2596} \quad .13 \\
\cancel{1} \quad \cancel{5192} \quad \cancel{.65} \\
\dfrac{\cancel{3}\times\cancel{20768}\times\cancel{3.25}}{\cancel{4}\times\cancel{3}\times\cancel{7500}} = \dfrac{1\times1298\times.13}{1\times1\times75} = \dfrac{168.74}{75} \\
\quad 1 \quad 1 \quad \cancel{3750} \\
\qquad\qquad \cancel{1875} \\
\qquad\qquad \cancel{375} \\
\qquad\qquad 75
\end{array}$$

The above cancellation was done exactly as explained in Section 2 (Factoring and Cancellation).

Now our equation looks like this

$$a=\sqrt{\frac{168.74}{75}}$$

Next divide 168.74 by 75. The result is 2.249. Then our equation becomes

$$a=\sqrt{2.249}$$

Finally find the square root of 2.249, which is 1.49.

PRACTICAL MATHEMATICS

Section 11

Lesson 1

For Step 1, consider the masses of facts and statistics that are constantly used in manufacturing establishments of all kinds, in engineering, in all commercial activities. For Step 2, learn that graphs provide an efficient means of comparing these figures and data. For Step 3, learn the general structure of graphs. For Step 4, study the two graphs (Figs. 1 and 2) used as illustrations.

GRAPHS

Everyone is more or less familiar with the drawings which architects and contractors use to show details for the different parts of buildings and to show the relation of the parts to one another. The machinist makes complicated pieces of mechanism easily, with only plans, elevations, and sections to follow.

There is another method of presenting in picture form the relations of quantities. This method is called **graphing.** While it is not so well known as the other methods, it is becoming more and more important, in some fields of activity, and is now often called the "shorthand" of the engineer, architect, machinist, and business man. Daily newspapers frequently devote space to graphs which show business conditions.

The advantage of graphs is that a person can see certain conditions at a glance, while it would take him considerable time to get the same information in writing. Sometimes graphs are drawn which require hours to calculate, but the observer can see the results in a finished graph in a few moments.

The range of application of graphs is almost unlimited and a knowledge of how to interpret and construct them should be a part of everyone's mathematical working material. A number of their uses with details of construction and meanings are given in this book.

In studying Lessons 1 and 2 of this Section, a student need not concern himself as to the method or reasoning employed in drawing the curves. It is not necessary for him to try to understand just how the given data is translated into the curve on the paper.

The purpose of these two lessons is to present the general construction of a graph and to impress on the student the economy of time and effort effected by the use of curves in preference to masses of statistical data or lengthy explanations in writing concerning the information that is to be conveyed.

LINE GRAPHS OR CURVES

General Structure. Refer to Fig. 1. The background of the figure shows a number of vertical and horizontal lines which cross each other. These lines are all equal distances apart, so that many small squares are formed. Notice that certain of the lines are heavier than others. These heavier lines form larger squares and there are five of the smaller spaces between the heavy lines. Paper marked off in this manner is called coordinate or cross-section paper. It can be obtained in various sizes and with the spaces of different lengths, according to the purpose for which it is to be used. The statistics and other facts are represented by these spaces.

In making a graph there must be chosen one vertical and one horizontal line to serve as a basis from which to count the required number of spaces. Each of these two lines is called an **axis.** The vertical axis is called an **axis of ordinates** and the horizontal axis is called an **axis of abscissas.** When the two axes are spoken of, they are called **coordinate axes.** Any convenient horizontal and vertical lines may be chosen as axes, but in all the illustrations used in this text, the outside lower and left lines on the cross-section paper (as 0X and 0Y in Fig. 1) are used as the axes. The point where these two axes meet is called the **point of origin** (marked 0 in Fig. 1).

The spaces on the coordinate paper are assigned values according to the data supplied in the problem. It is evident then that these values will vary greatly and must be chosen to suit the requirements of the problem with which we are working. The data or figures given in the problem must be closely observed in order that suitable values may be given the spaces. No set rule can be stated.

The student should note here that the term "curve" refers to the lines drawn to illustrate the data, whether such lines prove to be straight lines, curved lines, or broken lines (a series of straight lines). In graphing, the word "curve" is applied to any line plotted on the cross-section paper from data supplied.

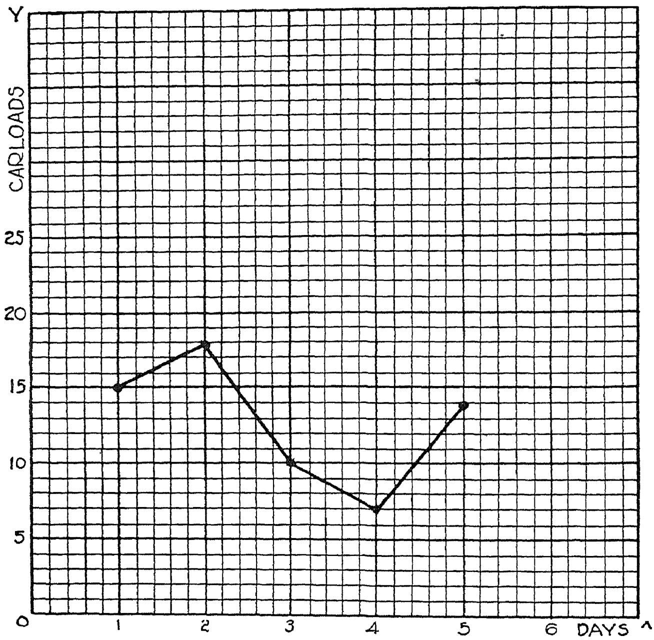


Fig. 1. Amounts of Coal Used by a Plant on Five Consecutive Days

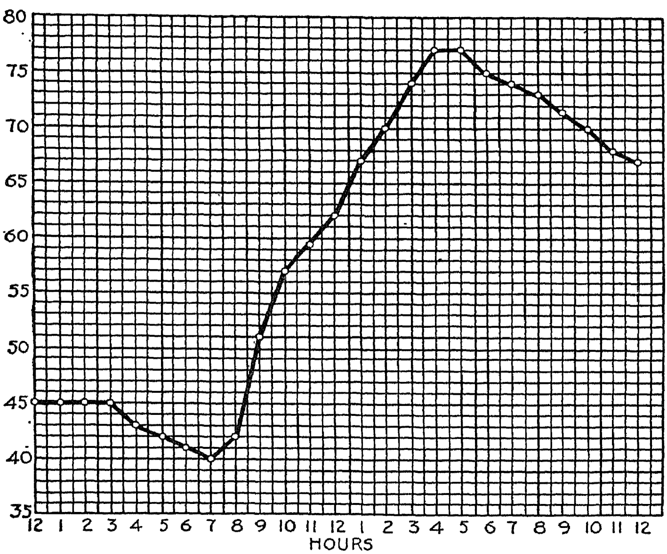


Fig. 2. Temperature Readings over a Twenty-Four Hour Period

Fig. 1 is a simple graph illustrating the following: A certain manufacturing plant used these amounts of coal on five consecutive days; first day, 15 carloads; second day, 18 carloads; third day, 10 carloads; fourth day, 7 carloads; and fifth day, 14 carloads. You will notice that each small space on the vertical axis has the value of 1 and each large space on the horizontal axis the value of 1. An executive of this plant could see by a glance at the curve just how much coal is being consumed each day and how the amounts vary.

Weather Bureau reports afford an interesting illustration of the use of graphs. This bureau often sends out curves showing the various temperatures over certain periods of time. Fig. 2 shows graphically the temperatures for twenty-four hours (from midnight of one day to midnight of the following day).

The horizontal axis represents the hours and the vertical axis the degrees of heat. The point of origin is not at 0 here but at 35, as there is no need for a lower value for the degrees than 35. Notice that it takes two spaces on the horizontal axis to represent one hour, and that each small space on the vertical axis represents one degree.

In order to plot this line, it was necessary to have a list of the temperatures for all of the twenty-four hours. It would take several minutes to read all this data and from it to compare the temperatures at various hours, but one glance at the line graph shows the movement and range of temperature for the entire time.

Other unit values for the spaces might have been used just as correctly, but values must be chosen that will suit the size of the paper used and display the line to the best advantage.

Lesson 2

For Step 1, bear in mind the use or value of graphs as a ready means of obtaining information. For Step 2, learn how to read line graphs. For Step 3, study examples 1 to 3. For Step 4, read the graphs illustrated in Figs. 6 to 8.

MEANING OF CURVES OR LINE GRAPHS

It is evident from the preceding discussion that each point is located by using two known quantities, one to be located on a vertical line and one on a horizontal line. The line obtained by connecting the various points thus located reveals the information contained in the given written or tabulated data.

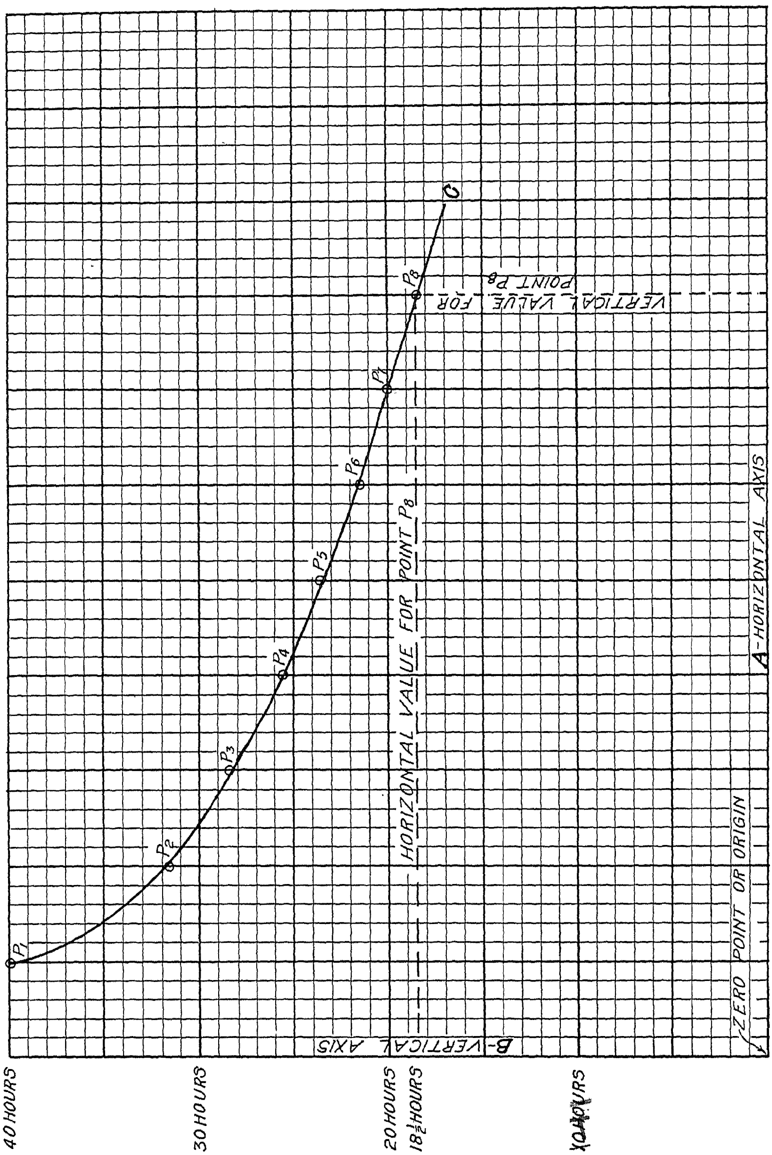


Fig. 3. Hours of Study Required to Complete Each of Eight Textbooks

Example 1. A student kept track of the study hours he spent on the eight books of his course and drew the curve shown in Fig. 3. Notice that the curve drops toward the horizontal axis. This drop indicates that it took the student less time to complete each succeeding text. In other words, this curve is a picture of the student's efficiency since he began to study. He grasped the material in each succeeding book more easily than in the preceding one.

Example 2. A certain steam railroad carrying suburban traffic was open for business and received an increasing amount of business up to the fourteenth year, when there was a sudden drop. The

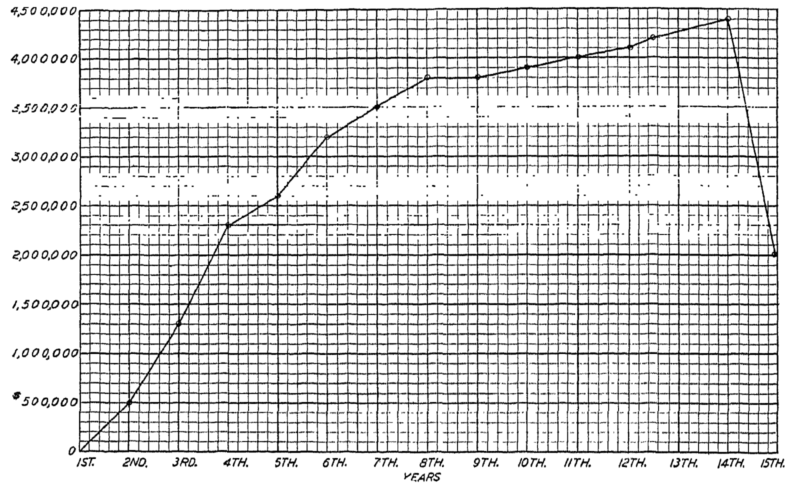


Fig. 4. Railroad Earnings over Fifteen-Year Period

Manager of the road was requested by the Board of Directors to present a record showing the gross income of the road for each year of operation. He presented the curve shown in Fig. 4.

The coordinates show years laid off on the horizontal axis and incomes on the vertical axis. The curve shows that the income per year increased each year up to the fourteenth year when it dropped abruptly. The cause was readily traceable to the influence of a new parallel electric line. As this new road made regular trips throughout the day and much more frequently than the steam road, it received most of the suburban business. This curve shows the relative value

of the yearly income much more clearly and strikingly than many pages of statistics.

Those of our students who are studying the various engineering courses will find many problems which can be illustrated by curves. The following is a practical one in Mechanical Engineering.

Example 3. The engineering department of a certain firm manufacturing transmission steel rope, having made tests of the horsepower developed by a rope running at certain velocities, expressed this relation in the form of a curve as shown in Fig. 5. The

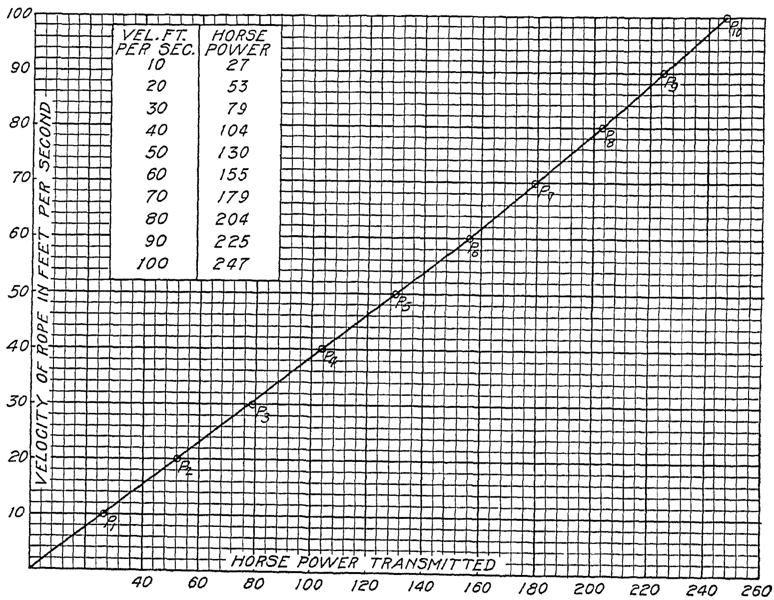


Fig. 5. Horsepower Developed by Transmission Steel Rope

points were located and the curve drawn by using the data arranged in tabular form at the upper left of the coordinate paper.

The curve is practically a straight line. This indicates that as the velocity increases, the horsepower increases in almost the same proportion, which information is far more readily obtained from a glance at the curve than from studying the mass of facts used in plotting the curve. Curves of this kind, therefore, save much time to the person who wants this certain information.

Example 4. A practical application of graphing is expressing the relation of numbers to the squares of those numbers. Fig. 6 shows a curve that has been plotted from using the numbers 1 to 12 and their squares.

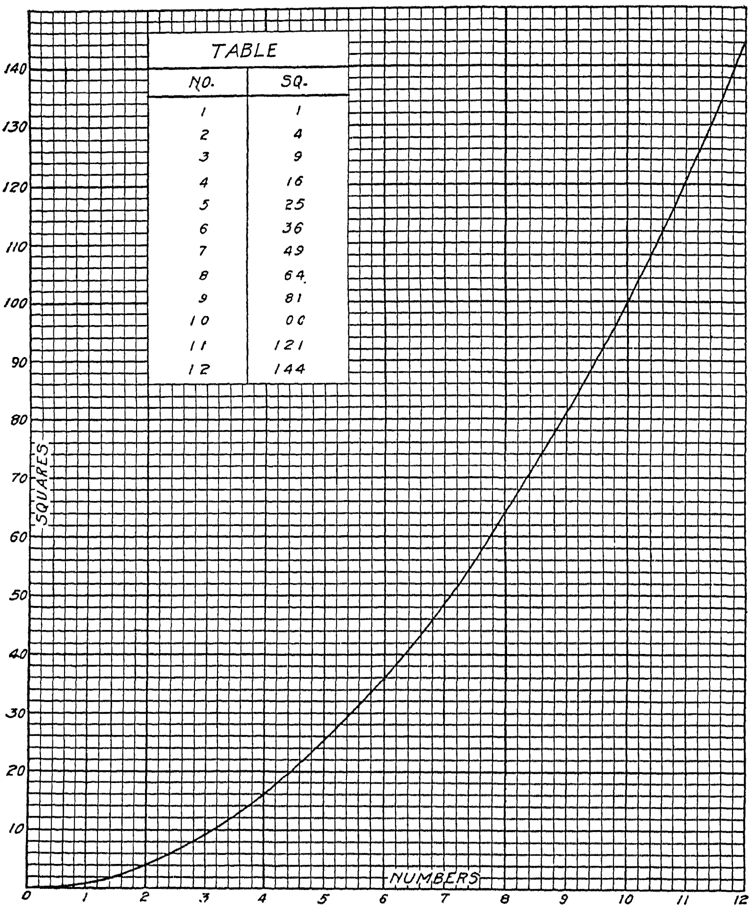


Fig. 6. Curve of Squares and Square Roots

The abscissas are the numbers 1 to 12 and the ordinates are the numbers 1 to 150.

A further discussion showing the usefulness of this curve is given in the next lesson.

Example 5. In Fig. 7 we have a graph showing two curves, each representing different data. The heavy black line shows the sharp upward turn of sales soon after the firm began advertising at the beginning of 1924. The curve representing the advertising

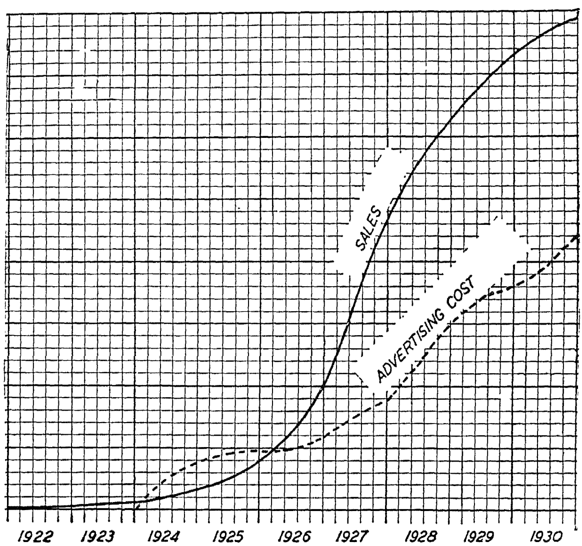


Fig. 7. Sales and Advertising Costs over Nine-Year Period

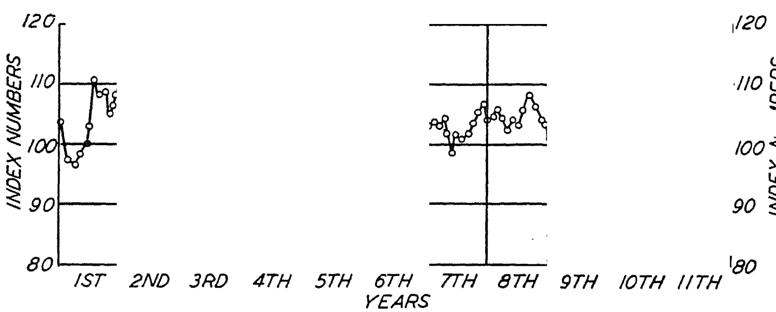


Fig. 8. Index of Business Activity in United States for Eleven-Year Period

expense is an upward curve also but not at nearly the same rate. The comparison of these two lines shows at a glance that the sales increased more rapidly and to greater amount than the outlay for advertising. Obviously it is much easier and simpler for a member

of the firm to get this information from the graph than from a tedious comparison of many figures.

Example 6. An erratic curve is shown in Fig. 8, giving the combined index of business activities in the United States for a period of years. The abscissas are the years and the ordinates are index numbers. From this curve it is immediately seen that business activity was the lowest in the third year and highest in the fifth year. It dropped quite low in the sixth year and from the end of the ninth year period it rose steadily.

Compare the time taken to get this information from the graph with the time it would take to obtain it from a mass of figures or statistics. This curve is a striking example of graphs as time savers.

Lesson 3

For Step 1, recall the use of graphs and what you have learned about reading line graphs. For Step 2, learn how to draw line graphs by working Illustrative Examples 1 to 6. For Step 3, study the method of obtaining required quantities from a given graph by studying Illustrative Examples 7 and 8. For Step 4, work the Practice Problems.

DRAWING LINE GRAPHS*

In the two previous lessons, various curves have been presented to the student and he has learned how to interpret their meaning and to appreciate their effectiveness. We have now come to the point in our study where only the data is given, and the student is himself required to construct the curve that will correctly convey the information in the data supplied.

There is an infinite variety of facts that can be conveyed to the observer by means of curves, so it would be entirely impossible to illustrate even a small percentage of them. However, several typical ones are explained and the principles used in them will serve as a basis for others that may be later required.

The student should provide himself with coordinate paper, which may be obtained at any office supply or stationery store, preparatory to the making of graphs and drawing of line curves.

ILLUSTRATIVE EXAMPLES

1. The prices of a certain commodity on the first of each month for a period of eight months were respectively: 34¢, 27¢, 19¢, 18¢, 18¢, 10¢, 27¢, and 30¢. Illustrate these facts by a curve.

Solution

Step 1. To assign spaces on the coordinate paper that will conveniently represent the given values. The two quantities involved are months and cents. Since the number of months is not large (ranging only from 1 to 8) we can let one large space represent one month. It does not matter whether you mark these on the vertical axis or on the horizontal. We have chosen the horizontal. (Fig. 9.)

Fig. 9. Commodity Prices at First of Each Month over Period of Eight Months

The numbers of cents range from 10 to 34, so it would hardly be practical to let one large space represent one cent as that would necessitate having 24 large spaces, making the graph rather unwieldy. So let one small space represent 1 cent, marking the values at 5-space intervals as shown.

Step 2. To locate the points. Referring to the given data we see that the price in the first month was 34 cents. Place your pencil

at the point marking the first month (at 1 on horizontal axis) and move up from there along the vertical line till you reach the horizontal line opposite the point marking 34 cents on the vertical axis. Mark it by a dot. (P_1 on graph.)

The next data given is that the price in the second month was 27 cents. Now place your pencil at the point marking the second month (at 2 on horizontal axis) and move it up along the vertical line till you come to the horizontal line opposite the point marking 27 cents on the vertical axis. Mark it by a dot. (P_2 on graph.)

In exactly the same manner locate the other points from the remaining data given in the problem. There will be eight points in all.

Step 3. Draw the curve by connecting the eight points in order.

2. Record was kept of the number of inches of rainfall for each month of the year. The record showed the following:

Rainfall for January	2.08 inches
Rainfall for February	2.30 inches
Rainfall for March	2.62 inches
Rainfall for April	2.72 inches
Rainfall for May	3.53 inches
Rainfall for June	3.42 inches
Rainfall for July	3.52 inches
Rainfall for August	3.02 inches
Rainfall for September	3.06 inches
Rainfall for October	2.43 inches
Rainfall for November	2.52 inches
Rainfall for December	2.05 inches

Express these facts by a curve.

Solution

Step 1. Choose values for the spaces that will conveniently represent the quantities in the problem. The two quantities are inches and months, the number of months being 12, and the inches varying from a little over 2 to less than 4. Since both these quantities involve small numbers, it will be all right to use one large space to represent each month and to use two large spaces (or ten small ones) to mark one inch of rainfall. Since there are decimal numbers involved in them, it will help to bear in mind that each small space represents one-tenth of an inch of rainfall. Mark the spaces for the inches as shown. (Fig. 10.)

Step 2. To locate the points. The first fact to be recorded on the graph is that the rainfall in January was 2.08 inches. Place your pencil at point 0 on the horizontal line (it also marks January) and move it up along the vertical line from there till you reach the point 2.08. Each small space represents one-tenth and since .08 equals .8 tenths, the point will be $\frac{8}{10}$ of a space above the 2. Mark it by a dot.

Next, show the fact that the rainfall for February is 2.30 inches. Start at the point marking February on the horizontal line and move

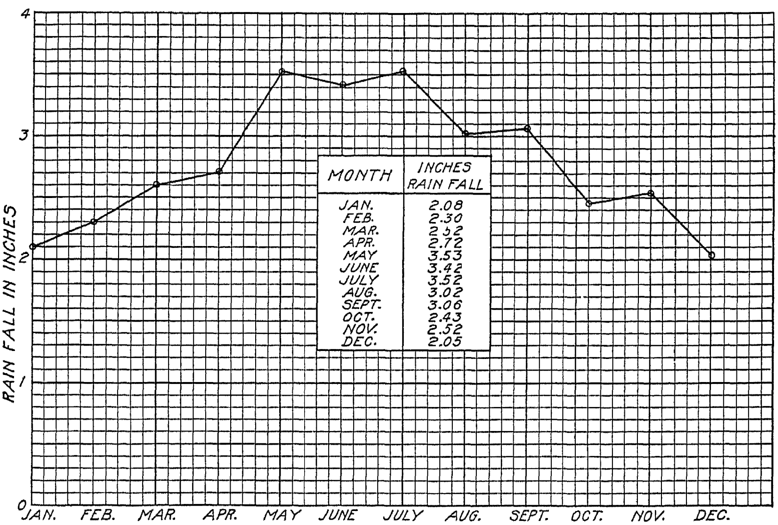


Fig. 10. Annual Mean Rainfall

up along the vertical line from there till you come to the horizontal line that would be opposite 2.3 on the vertical axis. Since .3 is three-tenths, the point is right on the third line above the line marking 2 inches. Mark this point by a dot.

Similarly locate the point marking March rainfall, 2.62 inches. It will be just a little more than six small spaces above the horizontal line marking 2 inches. In just the same way find the other points, marking each by a dot as you find it.

Step 3. Join the points in order by lines. Your finished graph shows at once the range of rainfall for the year and how the rainfall in any one month compares with that of any other month.

3. Show by a line curve the following data. The current loads used by an electric train, for certain hours of the day were:

At 12 o'clock	37,500 amperes
At 4 A.M. o'clock	10,500 amperes
At 9 A.M. o'clock	28,000 amperes
At 1 P.M. o'clock	45,000 amperes
At 3 P.M. o'clock	78,000 amperes
At 4 P.M. o'clock	75,000 amperes
At 5 P.M. o'clock	80,000 amperes
At 6 P.M. o'clock	73,500 amperes
At 8 P.M. o'clock	78 500 amperes
At 12 midnight	48,500 amperes

Solution

Step 1. Mark off the unit spaces on the vertical and horizontal axes. The two quantities are hours and amperes. The hours mentioned in the problem cover a period of twenty-four hours, so it

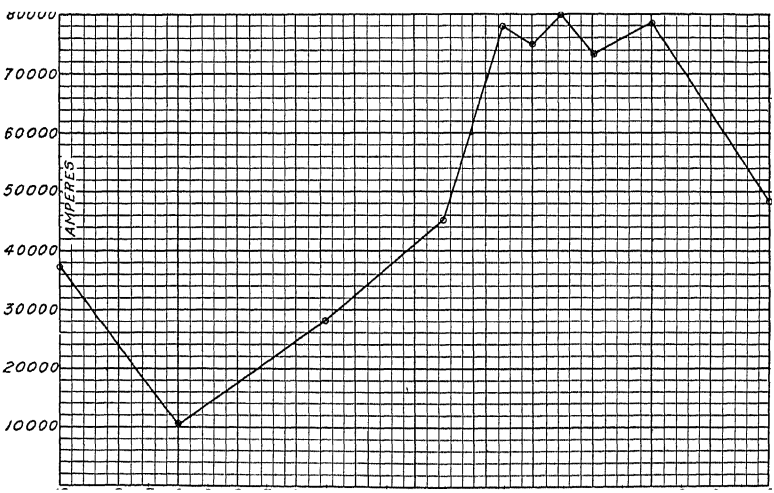


Fig. 11. Current Consumption of Electric Road for Twenty-Four Hour Period

would not be well to let one large space represent one hour, and to have one small space representing one hour would make the graph too crowded. So let each large space represent two hours. The hours are named between 12 midnight and noon, then to midnight again. (Fig. 11.)

The various loads are given in thousands of amperes, so it is evident that we cannot use single amperes as units, the numbers

are too large. Mark off the graph, letting each large space represent 10 thousand amperes. Thus, each small space will mark 2 thousand amperes, and half a small space will mark 1 thousand amperes.

Step 2. Locate the points representing the load for the various hours given in the data just as you did in Problems 1 and 2. Notice that the 12 o'clock load is 37,500 or $37\frac{1}{2}$ thousand amperes, and the 4 A.M. load is $10\frac{1}{2}$ thousand amperes. In locating the point for the 9 A.M. load, you will see that the point marking 9 A.M. comes midway between two vertical lines, so follow up between those two lines till you come to the line opposite the point marking 28 thousand on the vertical axis. Similarly with the 3 P.M. and 5 P.M. loads.

Step 3. Connect the points in order by lines, and you have a graph that shows immediately at what hour the maximum load and the minimum load were carried.

4. Plot the curve from the following data for the cost of high-speed engines.

Dollars		Horsepower
6,400	....	400
9,600	....	600
12,800	....	800
16,000	....	1,000
24,000	....	1,500
32,000	....	2,000
40,000	....	2,500
48,000	....	3,000

Solution

Step 1. Assign values to spaces on graph. Both the quantities in the problem involve large numbers, so we cannot make single spaces represent single units of the quantities. Let the horsepower be marked on the horizontal axis and the dollars on the vertical axis. Since the dollars are all in thousands, make the spaces mark thousands of dollars, and since the horsepower involves smaller numbers, let the spaces on the horizontal axis represent hundreds of horsepower. (Fig. 12.) You will note that a study of the quantities is always necessary to assign convenient space values to represent them.

Our numbers of dollars range from 6 thousand to 48 thousand. We have six large spaces on the graph paper to represent this range of 42 thousand. So if we let each large space be 10 thousand, we

can show all the amounts on the paper. Each small space will represent 2 thousand dollars.

The horsepower numbers range from 4 hundred to 30 hundred and we have 8 large spaces to represent this, so we can let each large space represent 5 hundred horsepower. Each small space will then represent 1 hundred horsepower.

Step 2. Locate the points. Referring to the data given, we see that an engine of 4 hundred horsepower costs 6.4 thousand dollars. Count 4 small spaces to the right of 0 along the horizontal

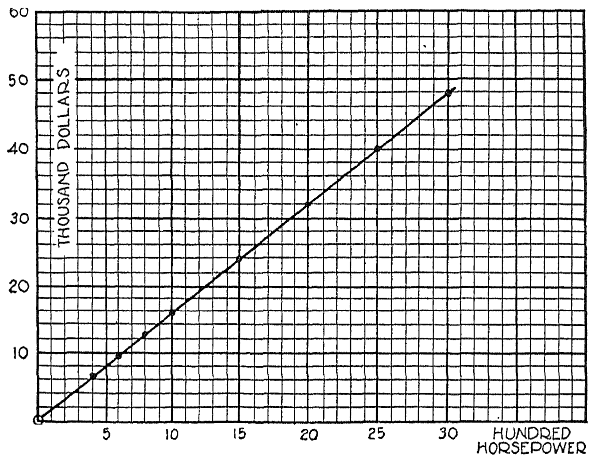


Fig. 12. Horsepower and Cost of High-Speed Engines

line (since each small space represents 1 hundred horsepower) and then up from there 3.2 small spaces (since each small space represents 2 thousand dollars). Mark the point by a dot.

Our next item in the given data is that a 600-horsepower engine costs 9.6 thousand dollars. So count 6 small spaces to the right of 0 and up from there 4.8 spaces (remembering that each small space represents 2 thousand dollars). Mark this point by a dot.

To locate the third point, count 8 small spaces to the right of 0 and up 6.4 (half of 12.8) and mark by a dot. Continue in this way until all the data is shown on the paper.

Step 3. Connect the points by lines. The curve in this case proves to be a straight line, showing that the cost increases in exact proportion to the horsepower of the engine.

5. Plot the curve irom the following data for the increase in a city's population for a period of 10 years.

Years	Population
1st	109,206
2nd	178,492
3rd	306,605
4th	400,000
5th	503,185
6th	655,000
7th	1,099,850
8th	1,335,813
9th	1,680,575
10th	1,949,116

Step 1. Select values to assign to the spaces on the graph. The marking of the years will give no difficulty as the number is

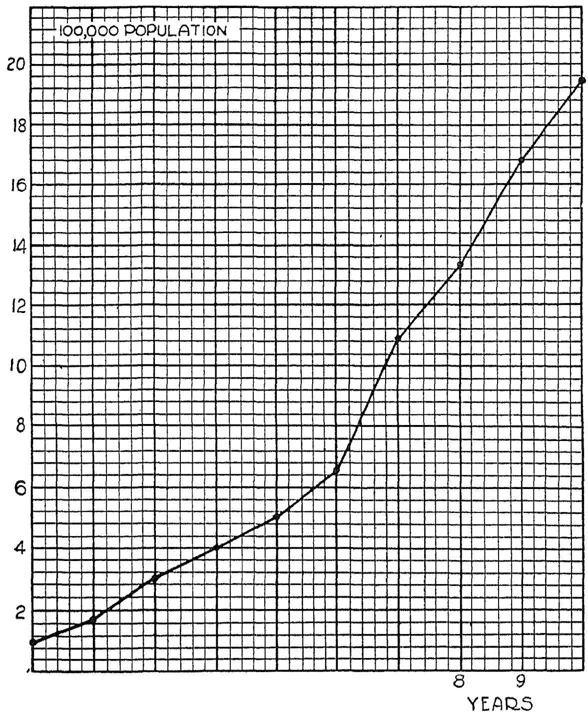


Fig. 13. Curve of Population for Ten-Year Period

small and each large space can represent one year. Since the numbers showing population are very large, we shall have to make each space represent a very large number. The numbers range from a

little over 109 thousand to almost 2000 thousand. We can hardly represent the numbers in thousands as the range from 109 to 2000 is still too large. So let us use hundred thousands as the unit value. This will give us a range of from about 1 hundred thousand to almost 20 hundred thousand. Twenty large spaces make a rather large graph, so let each large space represent 2 hundred thousand, or each small space represent 40 thousand. (Fig. 13.)

Step 2. Locate the points. The first fact in our given data is that in the first year the population was 109,206, a little over 1 hundred thousand. So count up on the line marking 1st year to just a little more than half one large space (remembering that one large space marks 2 hundred thousand). Mark the point by a dot.

Next, we have that the population was 178,492 or a little more than $1\frac{3}{4}$ hundred thousand. Then count up from the point marking 2nd year to just a little below the horizontal line marked 2. Mark the point by a dot.

The population in the third year was 306,605, just a little more than 3 hundred thousand. Halfway between the horizontal line marked 2 and the one marked 4 will be the location of the 3 hundred thousand point. Count up from the line marking the third year to a little past the horizontal position that marks the 3 hundred thousand line. Mark the point by a dot.

All the other points are located by a similar reasoning. For the 6th year the population was a little over $6\frac{1}{2}$ hundred thousand; in the 7th year, almost 11 hundred thousand; in the 9th year almost 17 hundred thousand. Locate each point just as accurately as you can.

Step 3. Connect the points in order by lines.

It is frequently necessary to make comparisons by drawing two or more curves on the same graph, as was illustrated in Fig. 7. The construction of such a group graph involves no new principle. Care must be taken that values are assigned to the spaces that will suitably represent all values in the various given sets of data.

6. The following figures give the number of telephone calls for each hour between 7 A.M. and 9 P.M. in the business district and in a residence district of a city. By means of two line curves on the same graph, show a comparison of these calls.

Hour	Calls in Business District	Calls in Residence District
7 A.M.	100	100
8 A.M.	200	700
9 A.M.	900	3,000
10 A.M.	6,000	6,800
11 A.M.	10,500	7,000
12 noon	10,000	6,400
1 P.M.	8,000	6,000
2 P.M.	7,400	4,500
3 P.M.	9,500	4,500
4 P.M.	8,500	4,000
5 P.M.	6,000	3,500
6 P.M.	3,400	3,400
7 P.M.	1,500	3,200
8 P.M.	400	3,000
9 P.M.	200	2,000

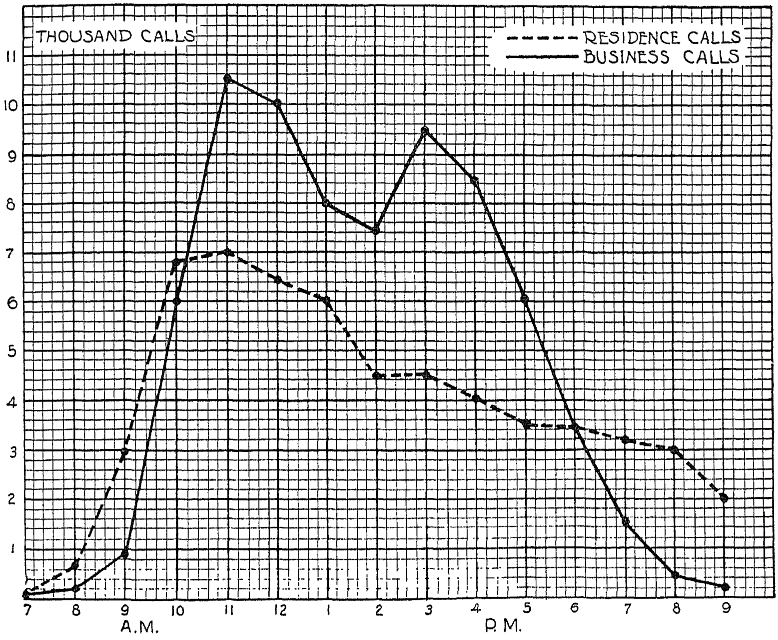


Fig. 14. Telephone Calls from City Business and Residence Districts for Period of Fourteen Hours

Solution

Step 1. Mark suitable values on spaces for the two quantities, which are hours and numbers of calls. The hours range from 7 A.M. to 9 P.M., a total of 15 different hours. Mark these on the horizontal axis. (Fig. 14.)

The numbers of calls show a range of from 200 to 10,500, so it will be best to use one thousand as the unit for the spaces on the vertical axis. We can then number the large spaces on that axis from 1 to 11.

Step 2. Locate the points for the first column of figures by the method we used in previous problems. Connect the points in order by lines.

Step 3. Locate the points for the second column of figures. Connect these points in order by a dotted line to distinguish it from the other curve.

Step 4. Put a note either on the graph or on the lines showing what each line represents.

7. Refer to Fig. 2. From the graph, find what the temperature was at 7 A.M. What was the temperature at 12 o'clock noon?

Solution

Step 1. Reading values from the graph in this way is the reverse of locating points by which to draw the curve. We are given the curve and one quantity and have to find the other quantity.

Place your pencil at the point marking 7 A.M. on the horizontal axis and move it up along the vertical line until the curve is reached. Then follow from there along the horizontal line toward the left until you come to the vertical axis, and note the value there. It is found to be 40. So the temperature at 7 A.M. was 40°.

Step 2. Follow up along the vertical line from the point marking 12 o'clock noon until you come to the curve. Then move along the horizontal line from there until you come to the vertical axis, and note the value. Since each small space represents 1°, the value is found to be 62°.

8. Refer to Fig. 11. From this graph, find what the current load was at 2 A.M.; at 9 P.M.

Solution

Step 1. Start at the point on the horizontal axis marking 2 A.M. and follow up the vertical line from there till the curve is

reached. Then follow the horizontal line to the left until you reach the vertical axis, and determine the value there. Each small space on the vertical axis represents 2 thousand amperes, so the required value is 24,000 amperes.

Step 2. Place your pencil at the point marking 9 P.M. Notice that this point comes midway between two vertical lines. Follow vertically from there until you reach the curve. The point where the curve is reached is between two horizontal lines. Follow to the left from there until the vertical axis is reached, which will be about one-quarter of a space above the point marking 70,000 amperes. This quarter space would represent $\frac{1}{4}$ of 2 thousand amperes, or 500 amperes. Adding this to 70,000 gives 70,500 as the required result.

9. Refer to Fig. 6. From this graph can be found without actual multiplying the square or square root of any number up to 12. Find the square of 6 and of 9.8 by reading the graph.

Solution

Step 1. Start at the number 6 on the horizontal axis and follow up that line until you come to the graph line. Move your pencil from this point over to the vertical axis and you will come to the point marking 36, which indicates that the square of 6 is 36.

Step 2. To find the square of 9.8 from the graph. The number 9.8 is not marked on the horizontal axis, but we can easily find its location. Since 5 small spaces represent 1 on this axis, each space represents $\frac{1}{5}$; and since .8 is $\frac{4}{5}$, the point marking 9.8 will be on the first line to the left of 10.

Now follow up this line till you come to the curve, and move from there over to the vertical axis, which is reached two spaces below the 100 point. This location will be 96 (remembering that each small space on the vertical axis equals 2 units). So the square of 9.8 is 96. You see how much shorter time this takes than actually squaring 9.8 by multiplication. The results are practically the same.

Try finding the square of other numbers from the graph and checking your results by Arithmetic.

10. Find, from the graph, Fig. 6, the square root of 120.

Solution

Step 1. Since finding the square root is the reverse of squaring a number, we can obtain the required square root by reversing the procedure in example 9. Locate the position of 120 on the vertical axis, follow along the horizontal line from there until the curve is reached and then move your pencil down to the horizontal axis. You will find that the reading there is just a trifle less than 11, which you can prove by Arithmetic to be the square root of 120. More accurate values could be found on larger scales.

PRACTICE PROBLEMS

1. (a) Plot the curve from the following data for the areas and diameters of circles using areas for vertical values and diameters for horizontal values.

Areas in Square Inches	Diameters in Inches
.7854	1
3.1416	2
7.0686	3
12.566	4
19.635	5
28.274	6
38.485	7
50.265	8

(b) What is the diameter of a circle having an area of 45 square inches, as read from your curve? Ans. 7.6-inches.

2. (a) Plot a smooth curve showing the water consumption per hour for a 400-kilowatt steam turbine by means of the values given below. Plot pounds of water per hour as vertical values and horsepower as horizontal values.

Pounds Water per Hour	Horsepower
5,000	300
7,500	500
11,000	750
13,000	875
17,500	1100

(b) From the curve determine how many pounds of water per hour will be required when the steam turbine is delivering 700 horsepower. Ans. Approximately 10,000 lbs.

3. The total tonnage passing through three seaports, A, B, and C, during each of four years was as follows:

Year	Tons through A	Tons through B	Tons through C
1st	13,500,000	12,200,000	4,500.000
2nd	16,800,000	15,900,000	4,500,000
3rd	24,500,000	25,500,000	8,500,000
4th	17,100,000	16,900,000	5,800,000

Make a comparison of these amounts by means of a three-line graph.

Lesson 4

For Step 1, recall the interpretation and making of line graphs. For Step 2, note the general structure of bar graphs. For Step 3, study the meaning of the graphs shown in Figs. 15 to 17. For Step 4, read the graph illustrated in Fig. 18.

BAR GRAPHS

In previous lessons we have studied how values may be indicated and comparisons made by line curves. There are various other ways of making graphs besides by lines. One of these ways is by drawing what are called **bar graphs.** Their purpose, of course, is that of all graphs, to present to the eye a diagram or picture that will give information quickly and vividly. Bar graphs are used very frequently, so it is well that you should know how to interpret their meaning and how to draw them from given data.

READING BAR GRAPHS

Example 1. In Fig. 15 we have a simple bar graph, showing the value of construction work done in a period of six years. The ordinates are in billions of dollars and the abscissas in years. The first column shows the number of billion dollars spent in construction the first year; the second column, the amount spent the second year; and so on.

A glance at the heights of the columns or bars gives an immediate comparison between the amounts spent in the different years. There was an increase every year over the previous one except in the fourth and fifth years when the amounts were equal.

Example 2. In Fig. 16 we have illustrated, by means of a bar graph, the relative amounts of money spent for candy during six consecutive years. The horizontal axis represents years and the vertical axis millions of dollars. The bars here are cross-hatched in different ways to make the contrast more vivid. The graph shows a great increase in the amount spent between the first and second year, then a decided drop in the third year, with a steady increase each year thereafter.

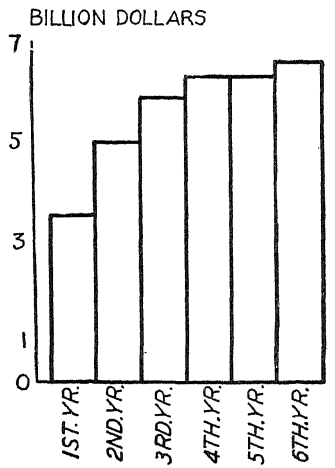


Fig. 15 Value of Construction Work over Six-Year Period

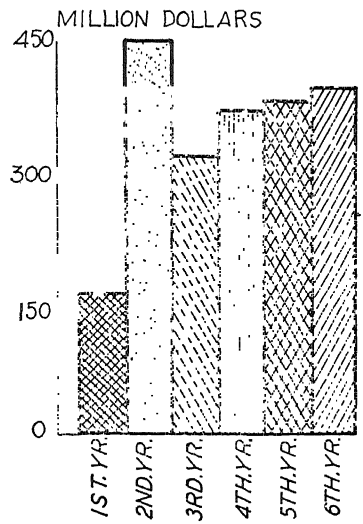


Fig. 16. Nation's Candy Bill over a Six-Year Period

Example 3. Fig. 17 illustrates a somewhat different type of bar graph. Here we have horizontal instead of vertical bars. These bars show the relative production of petroleum in various countries that are the leaders in producing that commodity. The lengths of the lines vary according to the number of barrels.

Example 4. An interesting graph is shown in Fig. 18. It gives a comparison of the output of lumber from five states in three different years, 1880, 1907, and 1922. The heavy black bar opposite the name of the state shows the percentage of output of lumber in that state in 1880; the lined bar represents the percentage output in 1907 and the plain bar the percentage for 1922. Such a drawing

is called a group bar graph. Note how the output has decreased in the first four states and has increased greatly in the last named state.

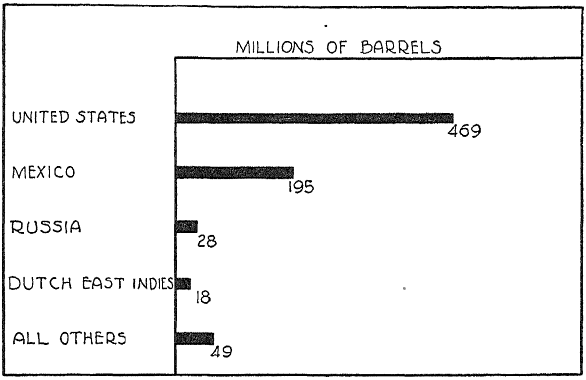
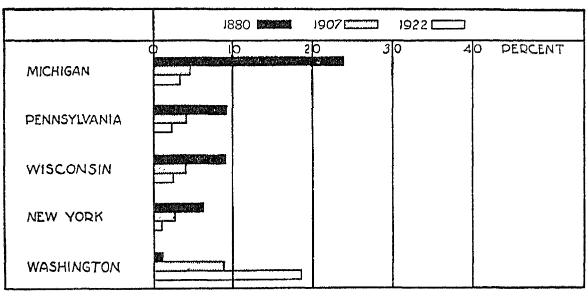


Fig. 17. Petroleum Output of Leading Countries



Fig, 18. Percentage Lumber Output of Five States for Three Different Years

It is at once evident how time and effort are saved by a study of the graph instead of the fifteen columns of figures that would be required to obtain the same information, showing once more the efficiency of graphs as a source of ready information.

Lesson 5

For Step 1, keep in mind the general form of bar graphs. For Step 2, learn how to draw simple bar graphs by studying Illustrative Examples 1 and 2. For Step 3, study Illustrative Example 3, to learn how to make a group bar graph. For Step 4, work the Practice Problems.

DRAWING BAR GRAPHS

In drawing bar graphs much the same principles are used as in making line graphs. Lay out a horizontal axis and a vertical axis and mark them according to the data furnished in the problem. Then draw bars of a length that will correspond to the amounts involved. As in drawing line graphs, no set rules for assigning values can be given as the data and amounts vary so much in different problems. Ease in making graphs of all kinds will come with continued practice.

ILLUSTRATIVE EXAMPLES

1. Represent by the use of a bar graph the following data which gives the output of automobiles for a period of six consecutive years.

Year	Automobiles
1st	4,000,000
2nd	3,500,000
3rd	4,100,000
4th	4,200,000
5th	3,400,000
6th	4,500,000

Solution

Step 1. Draw the two axes. The bars could be drawn either horizontally or vertically. We have drawn them vertically. On the horizontal axis mark off by dots a convenient space to represent the years. These spaces, of course, must be equal since each represents one year.

Since the numbers of cars are very large, we shall have to choose a large unit on thc vertical axis. One thousand cars can be conveniently used as a unit and each space on the vertical axis may represent 1000 cars. (Fig. 19.) The numbers of cars range from 3500 thousand to 4500 thousand, so 5 spaces representing one thousand each will be enough to mark off on the vertical axis.

Step 2. The first fact to be shown on the graph is that in the first year 4,000,000 cars (i.e. 4000 thousand) were produced. Therefore draw a bar so that its top will be just opposite the point marking 4000. The next fact is that in the second year 3,500,000 cars (3500 thousand) were made. Draw a bar so that its top will reach to a point just midway between the 3000 and the 4000 points.

Draw the other bars similarly, gauging their height to correspond to the number of thousand cars.

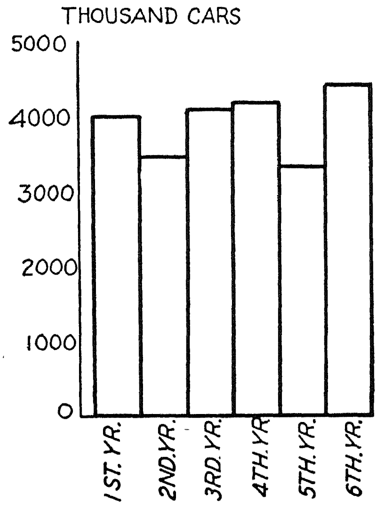


Fig. 19. Automobile Output in United States for Six-Year Period

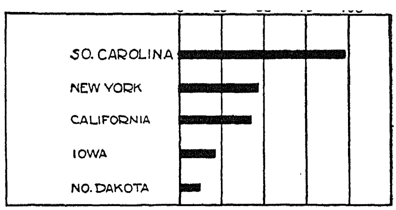


Fig. 20. Number of Persons Employed in Agriculture per Thousand Acres, in Five States

2. The number of persons employed in agriculture on each thousand acres of crop land in five states is as follows: in South Carolina, 98; in New York, 42; in California, 42; in Iowa, 20; in North Dakota, 12. Indicate these facts by a bar graph. (Fig. 20.)

Solution

Step 1. Draw the axes. It will be most suitable to draw horizontal bars here and to place the names of the states on the vertical axis so that they can be read without turning the page around. The numbers of people range from 10 to 98. To make the bars of convenient length, let each space on the horizontal axis represent

25, thus requiring four equal spaces. Mark them. (Other units, as 5 or 10, could be used if desired, depending on the size of the paper used.)

Step 2. Draw a horizontal bar that will correctly represent 98 on the graph. It will reach almost to the line marking 100. Next, draw a horizontal bar whose length will represent 42 on the graph. It will reach to a little farther than halfway between the lines marked 25 and 50. In the same manner, draw the other three required lines and the graph is completed.

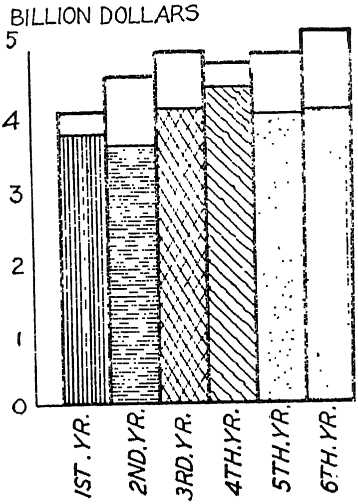


Fig. 21. Comparison of a Country's Imports and Exports over Six-Year Period

3. By means of a single bar graph for each year, show the value of the imports, exports, and balance of trade (the difference between exports and imports) of a certain country for a period of six years.

Year	Imports (Billion Dollars)	Exports (Billion Dollars)
1st	3.8	4.2
2nd	3.6	4.6
3rd	4.25	5
4th	4.5	4.75
5th	3.7	4.9
6th	4.1	5.25

Solution

Step 1. Draw the axis, marking the numbers of the years on the horizontal axes, and the values on the vertical axes. The values range up to 5.25 billion dollars so a little more than five spaces will be sufficient to mark them off, each space then marking one billion dollars. (Fig. 21.) If we represent the exports by the entire length of the bar and the imports by a shaded portion, the third quantity (the balance of trade) will be shown by the unshaded portion of the entire bar. Thus we shall have the three items indicated on a single bar.

Step 2. The value of the exports the first year was 4.2 billion dollars, therefore draw a bar so that its length will correspond to 4.2 spaces on the vertical axes.

The value of imports the first year was 3.8 billion dollars. To represent this fact, shade the bar you have drawn so that the shaded part will reach to the point marking 3.8 on the vertical axes. Draw the other four bars in exactly the same way, and the graph will be completed. The bars should be shaded in various ways so that they will stand out distinctly from one another.

PRACTICE PROBLEMS

1. Indicate the following data by a bar graph: A year's output of iron ore for four different countries was, respectively, 65,300,000 tons; 6,200,000 tons; 12,700,000 tons; 11,700,000 tons.

2. The yearly value of new buildings constructed and the value of buildings destroyed by fire for five consecutive years, were as follows:

Year	Value of New Buildings	Value of Buildings Destroyed by Fire
1st	$506,000,000	$216,000,000
2nd	430,000,000	239,000,000
3rd	611,000,000	202,000,000
4th	535,000,000	237,000,000
5th	535,000,000	235,000,000

Using a single bar to represent the facts pertaining to each year, compare the above facts by means of a bar graph.

Lesson 6

For Step 1, keep in mind the use or value of all graphical representation. For Step 2, note the structure of circle graphs by studying Fig. 22. For Step 3, study the Illustrative Example. For Step 4, work the Practice Problem.

CIRCLE GRAPHS

The circle is frequently used as a basis for drawing graphs, especially where the data involves percentage. The entire area of the circle represents 100 per cent and sectors of the circle are used to show the various percentages given in the data.

Example. Fig. 22 is such a graph. The entire circle represents the total amount spent in a certain district for school purposes. Note that the teachers' salaries took 51.4 per cent of the total and therefore this per cent is represented by a little more than half the circle.

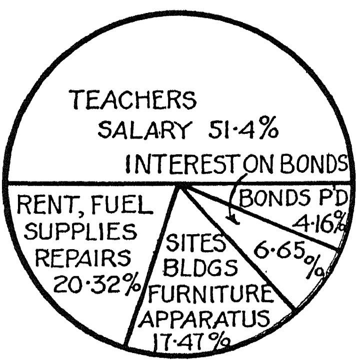


Fig. 22. Distribution of Revenue Raised for School Purposes

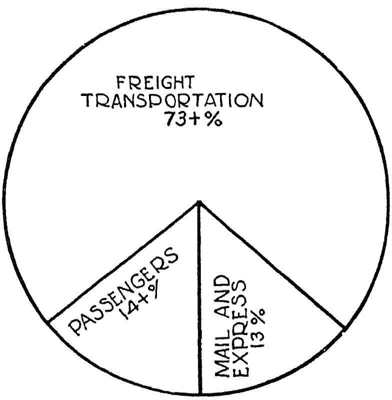


Fig. 23. Comparison of Sources of a Railroad's Income

The next largest item was for rent, fuel, repairs, and supplies (20.32%), represented by approximately one-fifth of the circle. Other percentages are marked by sectors of the circle according to their size.

In case amounts are given instead of percentages, it would be best to first find the total amount involved and then find what fraction or percentage each item forms of the total. The circle can then be marked off into sectors to represent these various percentages.

ILLUSTRATIVE EXAMPLE

1. The earnings of a railroad were as follows: Freight transportation, \$4,500,000; passengers, \$900,000; express and mail, \$800,000. Show this data by means of a circle graph.

Solution

Step 1. Find the total amount of earnings from all sources by adding the three items. This gives \$6,200,000.

Step 2. Find the percentage each amount forms of the whole:

Per cent of earnings from freight $(4{,}500{,}000 \div 6{,}200{,}000)$
$= .73+$ or $73+\%$

Per cent of earnings from passengers $(900{,}000 \div 6{,}200{,}000)$
$= .14+$ or $14+\%$

Per cent of earnings from express and mail $(800{,}000 \div 6{,}200{,}000)$
$= .13-$ or $13-\%$

Step 3. Mark off sectors of a circle that will correspond to these percentages. (Fig. 23.)

PRACTICE PROBLEM

1. A family income of \$2000 was spent as follows: Food, \$800; rent, \$250; clothing, \$270; fuel and light, \$90; insurance, \$45; the balance for sundries. Make a circle graph that will illustrate these expenditures.

GENERAL SUGGESTIONS FOR DRAWING GRAPHS

1. Draw the axes starting at the lower left-hand corner, the axis of ordinates vertically and the axis of abscissas horizontally.

2. Examine the given data, so as to be able to select suitable unit values for the spaces on the paper. These unit values will necessarily vary greatly. Notice the range between the largest and smallest quantity to be represented on each axis of the graph. Divide this difference by the number of available spaces and the result will give approximately the quantity each space must represent.

3. The point where the two axes connect need not necessarily be 0. The conditions of the individual problem must dictate whether 0 or some other number is more suitable.

4. When the data involves very large numbers or where decimals are involved, it is sometimes difficult to obtain perfect accuracy. However, great care should always be taken to get the points properly placed.

5. Make yourself adept in reading and drawing graphs by studying those you may encounter in your reading and drawing curves of data or statistics that may appeal to you.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on Page 39 (top folio).

If you miss more than two of the problems it means you should review the whole Section very carefully.

Do not try this trial examination until you have worked every practice problem in the Section.

Do not start the final examination until you have completed this trial examination.

Note: Graph paper is sold by all stationers; however, the student may rule his own if he chooses.

1. A family of two has an income of $500.00 per month. Their expenditures and other divisions of income are as follows:

Savings	$100.00
Food	70.00
Rent	75.00
Clothing	50.00
Operating Expense	125.00
Amusement	30.00
Insurance	50.00

Show this data by means of a circle graph.

2. From the following data plot the curve for the areas and diameters of circles, using areas for vertical values and diameters for horizontal values.

Areas, Sq. In.	Diameters, In.	Areas, Sq. In.	Diameters, In.
.7854	1	19.635	5
3.1416	2	28.274	6
7.0686	3	38.485	7
12.566	4	50.265	8

3. Using the answer to Problem 2, determine the diameter of a circle which has an area of 26 square inches.

4. The weather reports for the first 15 days of December showed maximum temperatures as follows. All are above zero. Make a line graph showing these conditions.

Date	Temp.	Date	Temp.
December 1	36°	December 9	23°
2	33°	10	31°
3	16°	11	36°
4	17°	12	30°
5	15°	13	32°
6	14°	14	25°
7	13°	15	23°
8	19°		

5. See sketch, Fig. 24.

Crosshatched bars = depth of snow.

Striped bars = length of time to fall.

Fig. 24 is a graph showing severe snowstorms which occurred in a certain city during seven different years. The bars show the depth and length of time the storms lasted.

(a) In what year did the greatest fall of snow occur?

(b) How much more snow fell in the 1929 storm than in the 1939 storm?

(c) In what year did the greatest amount of snow fall in the shortest length of time?

(d) What was the total snowfall in inches for the seven storms?

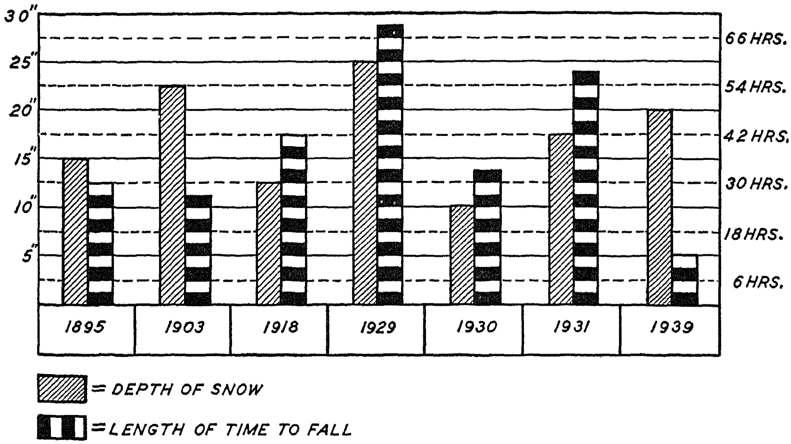


Fig. 24. Graph Showing Comparisons in Depth of Snowfall and Duration of Storms

EXAMINATION

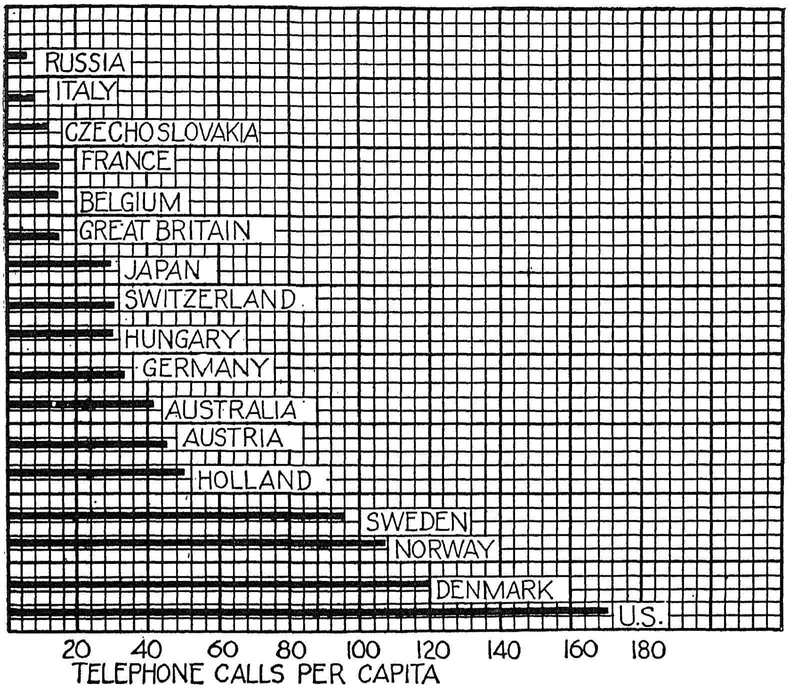


Fig. 25. Telephone Calls per Capita

1. Make a curve for the following temperature readings ranging hourly from 12 midnight to 12 noon. The first reading is for 12 midnight and the other readings are in order for each hour following. 2°, 5°, 9°, 15°, 14°, 16°, 18°, 20°, 24°, 26°, 28°, 32°, 35°. Use ordinary graph paper. One large square equals 5° and one large square also equals 1 hour.

2. Fig. 25 is a graph showing the number of telephone calls per capita in various countries of the world.

(a) How many calls per capita were made in Sweden? in Japan? in France?

(b) How many more calls were made in Holland than in Great Britain? How many times more calls in Norway than in Belgium?

3. The population of the United States by decades is given:

Year	Population
1st	3,929,000
11th	5,308,000
21st	7,229,000
31st	9,663,000
41st	12,806,000
51st	17,069,000
61st	23,191,000
71st	31,443,000
81st	38,558,000
91st	50,155,000
101st	62,669,000
111th	76,295,000

On graph paper or on paper ruled like graph paper, draw a curve that will correctly present the above data. The distance from one heavy line to another equals 10 million and also one decade.

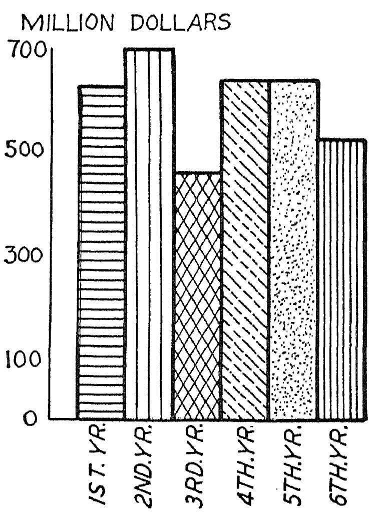


Fig. 26. Business Losses over Six-Year Period

4. Refer to Fig. 26. This bar graph represents business losses over a period of six years. Estimate (a) the amount lost in the third year; (b) the difference between the losses in the first and sixth years.

5. A certain manufacturing concern appropriated \$60,000 for running expenses for one year. Of this amount \$30,000 was to be spent for advertising, \$15,000 for salaries, and \$15,000 for materials and maintenance. Draw a circle graph which will show the proportionate allotment to each department.

NOTE:—If you have a compass draw the circle about two inches in diameter. Otherwise use a silver dollar, half dollar, or any other small round object which you can draw around to make a circle.

6. A man's income for seven years was respectively \$2,500, \$3,000, \$2,800, \$2,900, \$4,000, \$3,200, \$4,500. His living expenses for the same years were \$2,000, \$2,500, \$2,000, \$2,800, \$2,500, \$2,000, \$3,300. Draw a group line curve that will show a comparison be-

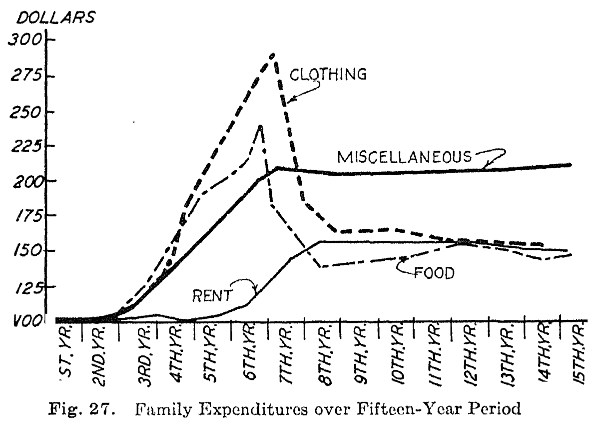


Fig. 27. Family Expenditures over Fifteen-Year Period

tween his earnings and expenses. Use ordinary graph paper. The side of each large square equals \$500 and the side of each large square equals also one year. Use full line for earnings and dotted line for expenses.

7. The group line graph shown in Fig. 27 represents the outlay of a family for clothing, rent, food, and miscellaneous expenditures over a fifteen-year period.

(a) In what year was the largest amount spent for food? What was that amount?

(b) In what year was the same amount spent for food, clothing, and rent? What was the amount?

(c) What does the graph indicate about the outlay for rent and miscellaneous expenditures after the eighth year?

8. In three different countries, A, B, and C, the percentages of cultivated land in rye, wheat, and potatoes are: A, rye 38%, wheat 5%, potatoes 18%; B, rye 15%, wheat 2%, potatoes 5%; C, rye 18%, wheat 35%, potatoes 18%. Draw a group bar graph, such as is illustrated in Fig. 18, to represent these facts. Make your graph so that the vertical lines are $\frac{1}{2}$ inch apart and let the distance between vertical lines equal 5%. Make the bars $\frac{1}{8}$ inch thick. Make the bar for rye solid black, the bar for wheat shaded, and the bar for potatoes blank.

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. First find the percentage of the whole income for each item of expense.

Savings	$100 ÷ 500	= 20%
Food	$70 ÷ 500	= 14%
Rent	$75 ÷ 500	= 15%
Clothing	$50 ÷ 500	= 10%
Operating	$125 ÷ 500	= 25%
Amusement	$30 ÷ 500	= 0%
Insurance	$50 ÷ 500	= 10%
Total Income		100%

Remember from previous sections that a circle contains 360° and this is to represent the total income. Each item of expenditure is a percentage of the income, and this same percentage applied to 360° indicates the sector of the circle that will represent this item. Therefore the next step is to find the various percentages of 360°.

Savings	20% of 360°	= 72°
Food	14% of 360°	= 50.4°
Rent	15% of 360°	= 54°
Clothing	10% of 360°	= 36°
Operating	25% of 360°	= 90°
Amusement	6% of 360°	= 21.6
Insurance	10% of 360°	= 36°
Total		= 360°

The final step is to mark off sectors in a circle equal to the number of degrees calculated. See Fig. 28.

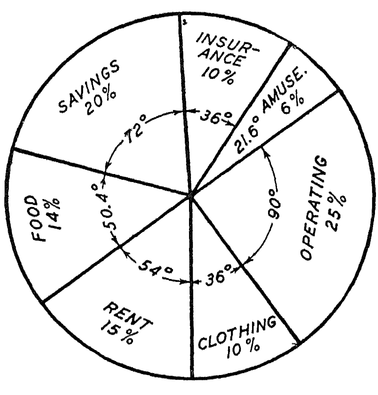


Fig. 28. Circle Graph Showing Divisions of Income

If you have a protractor you can solve this problem exactly as shown. Without such a drawing instrument you must simply approximate the various sectors in the circle.

2. In a problem of this kind having decimal numbers such as .7854, 3.1416, 7.0686, etc., it is impossible to lay them out on graph paper of ordinary size. Therefore, approximate them and call .7854 = 1, 3.1416 = 3, etc., as shown in the following:

Call .7854 = 1.	Call 19.635 = 20.
3.1416 = 3.	28.274 = 28.
7.0686 = 7.	38.485 = 38.
12.566 = 13.	50.265 = 50.

Note that any decimal less than .5 has been dropped. For any decimal over .5 we have added 1 to the whole number. Thus .7854 = 1, 3.1416 = 3, and 19.635 = 20.

The areas, which must be vertical values, are labeled in Fig. 29 at intervals of 10 sq. in. Thus starting at zero and counting 10 spaces upward, put *10.*

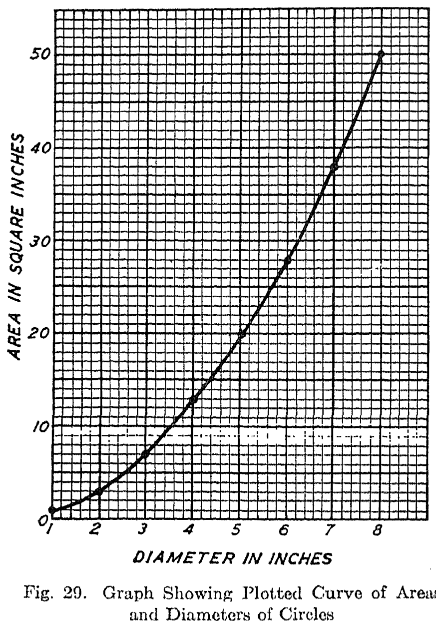


Fig. 29. Graph Showing Plotted Curve of Areas and Diameters of Circles

Then counting 10 more spaces above the *10,* put *20,* etc. For the diameters, which must be horizontal values, any number of spaces may be used to represent one inch, but by using 5 spaces as an inch the resulting curve is less steep and therefore more desirable.

Take the first area, namely .7854, and the first diameter (1 inch) for an example. The value .7854 was changed to 1. Thus, starting at zero in the vertical values, count up one space. Put a dot at this point because the vertical line is exactly above 1 for the diameter and the problem gives 1 inch as the diameter. Next, take 3.1416 sq. in. and diameter of 2 inches. The value 3.1416 was changed

to 3. From zero count upward 3 spaces. Then move horizontally to the right to the vertical line directly above 2 in the diameters. Put a dot at this point. The remaining part of the problem is done in exactly the same manner.

3. Count upward 6 spaces above the *20* (vertical value) in Fig. 29. Then follow this horizontal line until it intersects the curve. From the point of intersection drop an imaginary line straight downward to the horizontal values. The imaginary vertical line comes between the *5* and *6* (horizontal values) and is 4 spaces beyond the *5*. There are 5 spaces between *5* and *6*; thus each space has a value of .2. Then for 4 spaces beyond *5*, the value is 5.8. Therefore a circle having an area of 26 sq. in. has a diameter of 5.8 inches.

NOTE: This is an approximate diameter because of the approximations made in the original areas in the answer to Problem 2.

4. There are several ways in which this chart might be drawn. Fig. 30 shows one of the common ways of charting temperature differences from day to day.

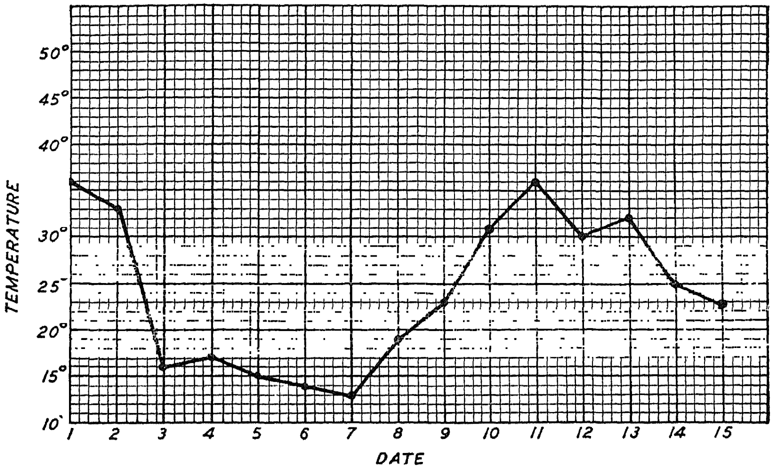


Fig. 30. Temperature Chart

The temperatures are shown as vertical values. The lowest maximum temperature is 13° so we can start with 10° and lay off the temperatures at 5° intervals, each small space having a value of 1 degree.

The days are shown as horizontal values; an interval of 5 spaces makes a pleasing graph. The student should always try to make graphs pleasing and easy to study. For example, if in Fig. 30 the days were only 2 spaces apart, the graph would look cramped.

Fig. 30 shows the correct answer.

5. In Fig. 24 the solid horizontal lines indicate the various amounts of snow in inches. The figures along the left side of the graph give the depths. The dotted lines represent the lengths of time in hours that the various storms

lasted. The hours are shown along the right side of the graph. The crosshatched bars represent amounts of snow while the striped bars represent lengths of time the various storms lasted.

(a) In a graph such as Fig. 24 the heights of the various crosshatched bars indicate the amounts of snow. For example, for 1929 the crosshatched bar is longer than any of the bars for the other years. Therefore the heaviest fall of snow occurred in 1929.

(b) In 1929 the total fall was 25 inches. In 1939 the total fall was 20 inches. Therefore 5 inches more snow fell in the 1929 storm than in the 1939 storm.

(c) One way to answer this question is to find the snow fall in inches per hour for the seven storms. To find the fall in inches per hour divide number of inches by the number of hours.

(1895) 15 inches divided by 30 hours equals $15 \div 30 = .5$ inch per hour.

(1903) $22\frac{1}{2}$ inches divided by 27 hours equals $22.5 \div 27 =$ approximately .83 inch per hour. It will be noted that the solid lines (which represent snow in the graph) are spaced so that the distance between any two such lines represents 5 inches of snow. The crosshatched bar for 1903 is half way between 20 and 25 inches, so it represents a fall of 20 inches plus one half of 5 inches, or $22\frac{1}{2}$ inches. The distance between any two of the dotted lines represents 12 hours. The striped bar for 1903 reaches about three-fourths of the distance between the dotted lines representing 18 and 30 hours. Thus this storm lasted 18 hours plus three-fourths of 12 hours, or $18+9=27$ hours.

(1918) $12\frac{1}{2} \div 42 = 12.5 \div 42 =$ approx. .29 inch per hour.

(1929) $25 \div 69 =$ approx. .36 inch per hour.

(1930) $10 \div 33 =$ approx. .30 inch per hour.

(1931) $17\frac{1}{2} \div 57 = 17.5 \div 57 =$ approx. .30 inch per hour.

(1939) $20 \div 12 =$ approx. 1.6 inches per hour.

The storm of 1939 with its fall of 1.6 inches of snow per hour answers this question.

(d) The total snowfall for all seven storms is as follows:

(1895)	15.0 inches
(1903)	22.5
(1918)	12.5
(1929)	25.0
(1930)	10.0
(1931)	17.5
(1939)	20.0
Total	122.5 inches

PRACTICAL MATHEMATICS

Section 12

MENSURATION—Part I

Lesson 1

For Step 1, keep in mind the fact that you are entering into a new subject and that it has terms and names which are new to you and which you must master, just as you must master a new vocabulary when you learn a new language. For Step 2, learn all the names and definitions. For Step 3, work Practice Problems 1 to 5. For Step 4, solve Practice Problems 6 to 8.

This text takes up problems in the study of Arithmetic that many students think are the most interesting of all. You will be required to apply many of the operations that you have already learned.

If you are not sure that you can handle fractions and decimals with ease and that you can extract square root accurately, then review those sections now so as to refresh your mind before beginning this text.

In the discussions in this book you should be free to think of the meaning of the different kinds of surfaces about which you will learn, see how the rules for measuring them are made, and why these rules are true. Then the actual work of applying the rules to the problems will be a pleasure.

As you advance, page after page, you will rejoice in your growing knowledge and the power it gives you to comprehend the higher steps of mathematical and scientific subjects. You will see that the study of Arithmetic is not an end in itself, but simply a means to an end—a key that helps you to understand the subjects that you will take up later in your course.

Mathematics has a language of its own, and certain signs and symbols belong to it. This subject more than any other uses signs and symbols. It is necessary for the student to know this language of signs and symbols and their uses, in order to understand thoroughly the textbooks on mathematics. Failure to learn the exact meaning

and use of the signs and symbols and of the definitions and terms used keeps many students from mastering mathematical subjects.

Mensuration is the process of computing the length of lines, the area of surfaces, and the volume of solids.

LINES

Lines are measured in linear units or units of length, as feet, yards, etc. Areas are measured in square units, as square feet, square yards, etc. Volumes are measured in cubic units, as cubic inches, cubic feet, cubic yards, etc.

A **line** has length only. (Fig. 1 (a).) It may be a boundary to a surface or intersection of two surfaces.

A **surface** has length and breadth (or width). (Fig. 1 (b).)

A **solid** has length, breadth, and thickness (or height). (Fig. 1 (c).)

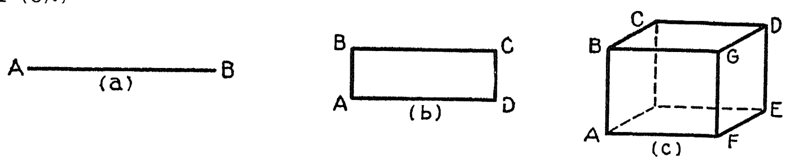


Fig. 1. (a) Line, (b) Surface, (c) Solid

These facts may be expressed in another way A line extends in one direction only; a surface extends in two directions; a solid extends in three directions. These directions of extension are spoken of as dimensions.

We think of a point as something that has position only, without length, breadth, or thickness. Thus, the end of a line is a point. The intersection of two lines is a point.

A **straight line** is one that has the same direction throughout its length. It is the shortest distance between two points. (Fig. 2 (a).)

A **curved line** is one that is continually changing its direction. It is sometimes called a curve. (Fig. 2 (b).)

A **broken line** is one made up of several straight lines. (Fig. 2 (c).)

A **plane surface** (or simply a plane) is a surface such that when a straightedge is applied to it, the straightedge will in every part touch the surface.

If two straight lines in the same plane are extended, they will meet or they will not meet. If they meet, they form angles. If they do not meet, they are parallel.

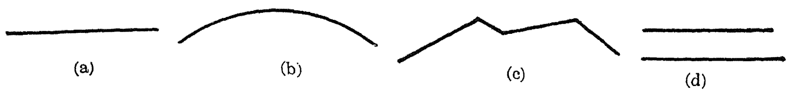


Fig. 2. (a) Straight Line, (b) Curved Line, (c) Broken Line, (d) Parallel Lines

Parallel lines are lines that lie in the same plane and are equally distant from each other at all points. (Fig. 2 (d).)

Lines are lettered to distinguish them. Thus, if one end is marked A and the other B, it is called the line AB or the line BA.

ANGLES

An **angle** is the amount of opening between two straight lines that meet at one point. The lines are called **sides** and the point of meeting is called the **vertex.**

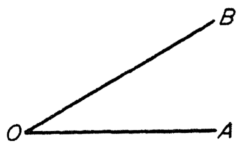


Fig. 3. Angle

Let the meeting point be the point O, as shown in Fig. 3. Now start at O and draw a straight line OA horizontally. Start again at the point O and draw another straight line OB, leaving a space between it and the line OA. This opening between the two lines is called an **angle.** The length of the sides makes no difference in the value of the angle.

Angles are usually named by placing a different letter at the end of each side and at the vertex. In naming the angle, the letter at the vertex must always be read second, that is, it must be read between the other two letters. Sometimes angles are indicated by numbers. (See Fig. 7.) The angle 2, using letters, would be named DOB or BOD. The angle 3 would be named DOA or AOD. Either way is correct.

The sign $\angle$ is often used in place of the word angle.

If one straight line meets another straight line so that the angles formed are equal, the lines are said to be **perpendicular** to each other and the angles formed are right angles and equal to 90° each. See Fig. 4 where four right angles are shown at the point B.

An **acute** angle is less than a right angle, as angle ABC in Fig. 5.

An **obtuse** angle is greater than a right angle, and less than two right angles, or less than 180° but over 90°, as angle ABC in Fig. 6.

The **bisector** of an angle is a straight line between the sides of the angle which cuts the angle into two equal parts.

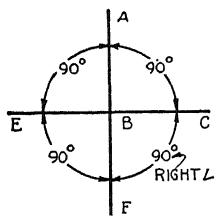


Fig. 4. Right Angles

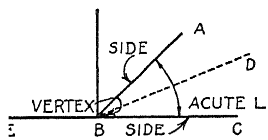


Fig. 5. Acute Angle

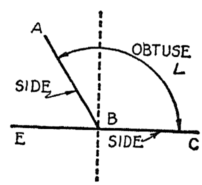


Fig. 6. Obtuse Ang

As two right angles are formed when one line meets another perpendicularly (Fig. 4), it follows that

(a) *The sum of all the angles about a point on one side of the straight line, such as EC, is equal to two right angles or* 180°.

If the line AB is extended to F, then it can be seen that

(b) *The sum of all the angles about any point such as B is equal to four right angles.*

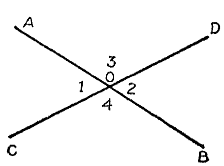


Fig. 7. Intersecting Lines

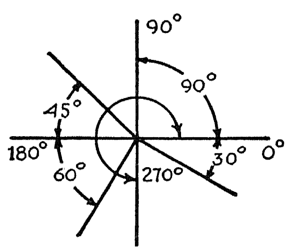


Fig. 8. Measurement of Angles

When two lines intersect, they form four angles, Fig. 7. The angles AOC and AOD are called **adjacent** angles because they have the same vertex O, and the line AO is common to both as it forms a side of each angle. From (a) it is seen that $\angle AOC + \angle AOD =$ two right angles. It may be proved by Geometry that the angles AOC

and DOB are equal to each other. These angles are called opposite or vertical. Therefore,

When two straight lines intersect, the opposite or vertical angles formed are equal to each other.

Take two narrow strips of cardboard or stiff paper and lay them on the table crossing each other. Fasten the point of crossing with a pin, so that the upper strip will turn freely. Set it first perpendicularly. You will see that you have made four right angles. Turn it in either direction. You will see that, no matter at what angle the one strip crosses the other, the sum of the four angles formed will equal four right angles.

Measurement of Angles. It has been shown in Fig. 7 that the sum of the angles about the point of intersection of two lines is always four right angles. Evidently other lines might be drawn through the same intersection, making the angles smaller and more in number, without changing the value of their sum. In measuring angles, therefore, it has been agreed to divide the four right angles about a point into 360 equal parts, called **degrees,** and indicated by the sign °. In this case, each of the four right angles would contain one-fourth of 360 degrees or 90°, while one-half,

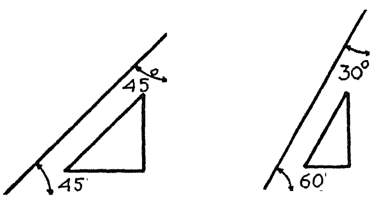


Fig. 9. 45° and 30°-60° Triangles

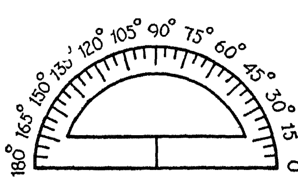


Fig. 10. Protractor

two-thirds, and one-third of a right angle would represent angles of 45°, 60°, and 30°, respectively, as shown in Fig. 8.

In drawing work, it will be found that the 45° triangle and the 30°-60° triangle, Fig. 9, are useful articles of equipment; but when it comes to accurate values of angles, some device like a protractor, Fig. 10, showing degrees and fractions of degrees, is necessary. Each degree is divided into 60 equal parts called minutes (′), and each minute into 60 equal parts called seconds (″). Thus, to represent an angle of 23 degrees, 47 minutes, and 9 seconds, it is written

23° 47′ 9″. The relations of these units may be summarized as follows:

60 seconds (″)	= 1 minute (′)
60 minutes	= 1 degree (°)
360 degrees	= 1 circle, or circumference
90 degrees	= 1 right angle
180 degrees	= 2 right angles, or a straight angle
360 degrees	= 4 right angles, or 1 circumference

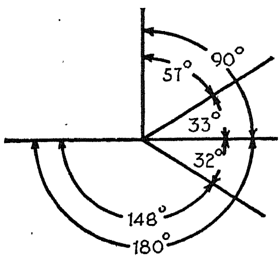


Fig. 11. Complementary and Supplementary Angles

Two angles are **complementary** when their sum is equal to one right angle or 90°; they are **supplementary** when their sum is equal to two right angles or 180°. Thus, an angle of 33° is the complement of one of 57° because 33°+57°=90°. Angles of 148° and 32° are supplementary because 148°+32°=180°. See Fig. 11.

The information conveyed in this lesson should be well understood so that it will be a part of your working material. Note especially how angles are named. Learn the meaning of complementary and supplementary angles, the definitions of the various kinds of angles, and the relations of the angles made when two lines intersect.

PROBLEMS FOR PRACTICE

1. How many seconds in 180 degrees? Ans. 648,000
2. How many right angles in 202° 30′? Ans. $2\frac{1}{4}$
3. What is the complement of 37°? Ans. 53°
4. What is the complement of 11° 47′ 3″? Ans. 78° 12′ 57″
5. What is the supplement of 131° 4′ 27″? Ans. 48° 55′ 33″
6. Does an acute angle contain more or less than 90°? An obtuse angle?
7. In Fig. 7 let angle *AOC* equal 52°; find the value of the three other angles about the point *O*.

Ans. ∠2=52°; ∠3=128°; ∠4=128°

8. How many seconds in 30′? Ans. 1800″

Lesson 2

For Step 1, think of the fact that you are going to study the names and forms of certain flat figures. These figures represent objects, such as the top of a table, the floor of a room, etc. For Step 2, learn all the new names, terms, and definitions treated in the lesson. For Step 3, study the Illustrative Examples. For Step 4, solve the Practice Problems.

PLANE FIGURES

A **plane figure** is a plane surface bounded by lines, either straight or curved.

The distance around a plane figure is called its **perimeter.** In Fig. 16 the perimeter $=2+3+2+3=10$.

The area of a plane figure is equal to the number of square units it contains. In Fig. 16 the area is 6 square units.

POLYGONS

A **polygon** is a plane figure bounded by straight lines. The boundary lines are called **sides** and the sum of the sides is called the **perimeter. Polygons** are classified according to the number of sides they have.

A **triangle** is a polygon with three sides. (Figs. 21 to 26.)

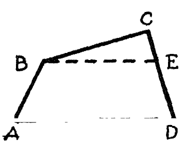


Fig. 12. Trapezium

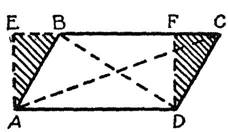


Fig. 13. Parallelogram

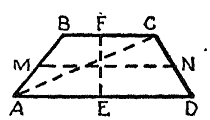


Fig. 14. Trapezoid

A **quadrilateral** is a polygon with four sides.

A **parallelogram** is a quadrilateral whose opposite sides are parallel and equal. (Fig. 13.)

A **trapezoid** is a quadrilateral having only two sides parallel. (Fig. 14.)

A **trapezium** is a quadrilateral having no two sides parallel. (Fig. 12.)

A **square** is a quadrilateral all of whose sides are equal and all of whose angles are right angles. (Fig. 15.)

A **rectangle** is a parallelogram whose angles are right angles. (Fig. 16.)

A **rhombus** is a parallelogram all of whose sides are equal but whose angles are not right angles. (Fig. 17.)

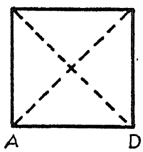


Fig. 15. Square

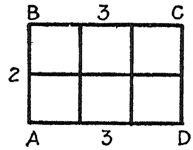


Fig. 16. Rectangle

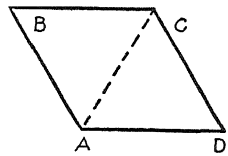


Fig. 17. Rhombus

A **pentagon** is a polygon with five sides. (Fig. 18.)
A **hexagon** is a polygon with six sides. (Fig. 19.)
An **octagon** is a polygon with eight sides. (Fig. 20.)

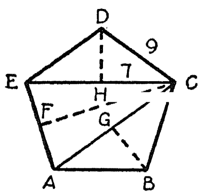


Fig. 18. Pentagon

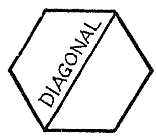


Fig. 19. Hexagon

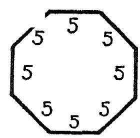


Fig. 20. Octagon

An **equilateral polygon** is one all of whose sides are equal. Figs. 15, 17, 18, 19, and 20 are illustrations.

An **equi-angular polygon** is one all of whose angles are equal. Such polygons are illustrated in Figs. 15, 16, 18, 19, and 20.

A **regular polygon** is one all of whose angles and all of whose sides are equal, as shown in Figs. 15, 18, 19, and 20.

A **diagonal** of a polygon is a line joining any two vertices (vertices is plural of vertex) that are not adjacent. Figs. 13, 14, 15, 17, 18, and 19 illustrate the diagonal. The diagonals of all parallelograms bisect each other at the center of the figure. Each also bisects the area. The diagonals of a square are equal and they bisect each other, forming right angles at the center. (Fig. 15.)

TRIANGLES

A **triangle** is a polygon enclosed by three straight lines called **sides.**

The **angles** of a triangle are the angles formed by the sides.

A **right-angled triangle,** often called a **right triangle,** Fig. 21, is one that has a right angle. The longest side (the one opposite the

right angle) is called the **hypotenuse,** and the other sides are sometimes called **legs.** When the two acute angles are equal, or 45° each, the legs are equal.

An **acute-angled triangle,** Fig. 22, is one that has all of its angles acute.

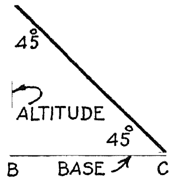


Fig. 21. Right Triangle

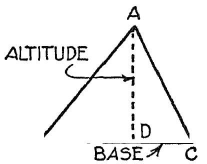


Fig. 22. Acute-Angled Triangle

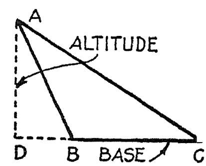


Fig. 23. Obtuse-Angled Triangle

An **obtuse-angled triangle,** Fig. 23, is one that has an obtuse angle.

An **equilateral triangle,** Fig. 24, is one that has all of its sides equal.

An **equiangular triangle,** Fig. 24, is one having all of its angles equal. It is also equilateral.

An **isosceles triangle,** Fig. 25, is one which has two of its sides equal. The angles opposite these two sides are equal.

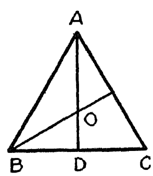


Fig. 24. Equiangular Triangle

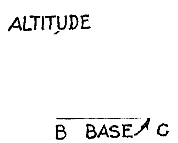


Fig. 25. Isosceles Triangle

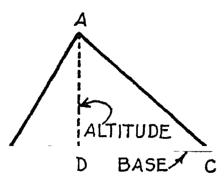


Fig. 26. Scalene Triangle

A **scalene triangle,** Fig. 26, is one which has no two of its sides equal.

The **altitude** of a triangle is the perpendicular drawn from the vertex to the base. In some figures, as in Fig. 23, it is necessary to extend the base so that the altitude may meet it.

The **base** of a triangle is the side upon which the triangle is supposed to stand. Any side may be taken as the base. In an isosceles triangle, the side which is not one of the equal sides is

usually considered the base. Triangles are named by reading the letters at each angle. The order in which the letters are read makes no difference. Thus the triangle, Fig. 21, could be named *ABC*, *ACB*, or *BAC*.

In a right triangle, one leg may be considered the base and the other the altitude, as in Fig. 21.

Cut a triangle from a piece of paper or cardboard and number the angles 1, 2, and 3. Cut off two points of the triangle and fit the three angles together. You will see that they make the equivalent of two right angles, or 180°, or a straight angle. This is illustrated in Fig. 27, and may be stated as a rule thus:

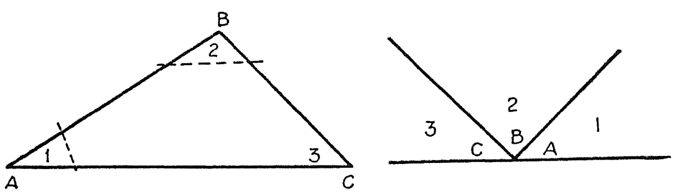


Fig. 27. Drawing Demonstrating Value of the Sum of the Angles of a Triangle

Rule. *The sum of the angles of any triangle is equal to two right angles, or* 180°.

Keep this rule in mind, for you will find it useful.

Knowing that the sum of all the angles of a triangle is equal to two right angles, or 180°, we can readily compute the third if two of the angles of a triangle are given.

ILLUSTRATIVE EXAMPLES

1. One of the angles of a right-angled triangle is 22° What is the third angle?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Since the triangle is right-angled, one of the angles is 90°. Find the sum of the two known angles	$90° + 22° = 112°$

Step 2	**Step 2**
The sum of all three angles is 180°, so to find the third angle, subtract the above sum from 180°	$180° - 112° = 68°$

68° Ans.

2. One of the angles of a triangle is 30°, another is 24° 27′. What is the third angle?

Solution

Instruction	*Operation*
Step 1	**Step 1**
First find the sum of the two given angles	$30° + 24°\ 27' = 54°\ 27'$
Step 2	**Step 2**
The sum of all three angles is 180°, so subtract 54° 27′ from 180° to find the third angle	$\begin{array}{rr} & 180° \\ & 54°\ 27' \\ & \\ & 179°\ 60' \\ \text{or} & \underline{54°\ 27'} \\ & 125°\ 33' \end{array}$

125° 33′ Ans.

PRACTICE PROBLEMS

The following numbers in each case represent two angles of a triangle. Find the size of the third angle.

1. 90° and 45° Ans. 45°
2. 90° and 60° Ans. 30°
3. 100° 30′ and 30° Ans. 49° 30′
4. 60° and 60° Ans. 60°
5. How would you describe the triangle of Problem 4?
Ans. Equiangular and equilateral

Lesson 3

For Step 1, recall Lessons 1 and 2. For Step 2, learn the law of the right triangle and how to apply it. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

LAW OF THE RIGHT TRIANGLE

If in the right triangle *ABC*, Fig. 28, the side *AB* is made 3 inches long, and the side *AC* is made 4 inches long, then the side *BC*, the hypotenuse, is found to be 5 inches long.

The proof of this may be found by considering Fig. 28. The square *ABED*, constructed on the side *AB*, contains 9 unit squares; the square *LCAM*, constructed on the side *AC*, contains 16 unit squares; and the square *CFGB*, constructed on the hypotenuse *BC*, contains 25 unit squares.

$$9 = 3^2$$
$$16 = 4^2$$
$$25 = 5^2$$
$$9 + 16 = 25$$
$$\text{or } 3^2 + 4^2 = 5^2$$

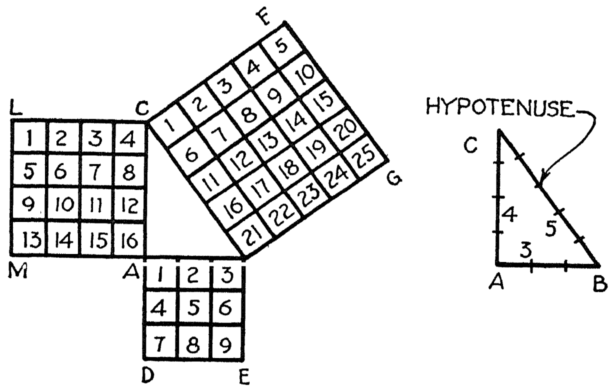


Fig. 28. Graphical Proof of the Law of the Right Triangle

Thus, if the squares of the two sides are added, the sum is the square of the hypotenuse.

Pythagoras, a Greek philosopher, who was born about 580 B.C., discovered this fact and so it has been called the Pythagorean (pronounced Pȳ′thag′o-re′an) Theorem, which is given below in rule form:

Rule. (*a*) *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

(*b*) *The square of one side equals the square of the hypotenuse minus the square of the other side.*

By these rules it is always possible to find the length of one side of a right triangle if the other two sides are known.

Rules (a) and (b) may be put in formula form as follows:

Let H represent the hypotenuse and let A represent one side and B represent the other side.

(1) $H^2=a^2+b^2$ (formula for Rule (a))

(2) $a^2=H^2-b^2$ (formula for Rule (b))

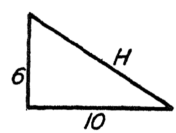
(3) $b^2=H^2-a^2$ (formula for Rule (b))

NOTE. In solving problems involving this rule, it is always very helpful to draw a diagram illustrating the triangle and mark on it the values given in the problem.

ILLUSTRATIVE EXAMPLES

1. One side of a right triangle is 6 inches and the other side is 10 inches. What is the length of the hypotenuse?

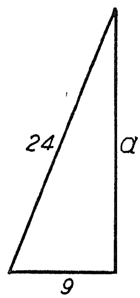
Solution

Instruction	*Operation*
Step 1 Draw the triangle, marking the three sides according to the information the problem gives	**Step 1** 
Step 2 We are to find the value of H. Our formula is $H^2=a^2+b^2$ Substitute in the formula the values for a and b, which we know	**Step 2** $H^2=a^2+b^2$ $H^2=6^2+10^2$
Step 3 Perform the indicated operations—square 6 and 10 and add the results	**Step 3** $H^2=36+100$ $H^2=136$
Step 4 If H^2 is 136, H is the square root of 136 11.66 in. Ans.	**Step 4** $H=\sqrt{136}=11.66$

NOTE. Observe carefully that we did not take the square root of 36 (which is 6) and the square root of 100 (which is 10) and call the result $6+10$, or 16. As you will recall from Section 10 on Formulas, we must perform all the operations indicated under the $\sqrt{\ }$ sign before we proceed to extract the square root. Thus, it was necessary first to add 36 and 100 and then obtain the square root of their sum.

2. The hypotenuse of a right triangle is 24 feet. One side is 9 feet long. What is the length of the third side?

Solution

Instruction	*Operation*
Step 1 Make a drawing of the triangle, marking the parts that are given in the problem. (It does not matter which letter, a or b, you use to represent the side to be found)	**Step 1** 
Step 2 We are required to find the value of a. The formula for finding a is $a^2=H^2-b^2$. Substitute the values the problem gives us for H and b	**Step 2** $a^2=H^2-b^2$ $a^2=24^2-9^2$
Step 3 Perform the indicated operations—find the squares of 24 and of 9 and subtract	**Step 3** $a^2=576-81$ $a^2=495$
Step 4 Take the square root of both sides of the equation 22.24 ft. Ans.	**Step 4** $a=\sqrt{495}=22.24$

3. A ladder 40 feet long, when placed on the ground 24 feet from the foot of a wall, just reaches the top of the wall. What is the height of the wall?

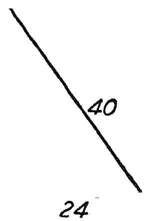
Solution

Instruction / *Operation*

Step 1

Draw a sketch illustrating the problem. Since the wall must be at right angles with the ground, we shall have a right triangle, of which the hypotenuse (H) will be the length of the ladder and the one side (a) will be the distance on the ground that the ladder is from the foot of the wall.

Step 1



Step 2

We are to find the value of b. Our formula for b is $b^2=H^2-a^2$. Substitute the values for H and a, that the problem gives

Step 2

$$b^2=H^2-a^2$$

$$b^2=40^2-24^2$$

Step 3

Perform the indicated operations

Step 3

$$b^2=1600-576$$
$$b^2=1024$$

Step 4

Take the square root of both sides of the formula

Height of wall is 32 ft. Ans.

Step 4

$$b=\sqrt{1024}=32$$

The following table of squares will be found useful for quick reference:

$11^2=121$	$15^2=225$	$19^2=361$	$23^2=529$
$12^2=144$	$16^2=256$	$20^2=400$	$24^2=576$
$13^2=169$	$17^2=289$	$21^2=441$	$25^2=625$
$14^2=196$	$18^2=324$	$22^2=484$	$26^2=676$

PRACTICE PROBLEMS

1. One side of a right triangle is 1 foot and the other side is 15 inches. What is the length of the hypotenuse? Ans. 19.2 inches

2. A rectangular park is 40 rods long and 30 rods wide. What is the length of a straight walk connecting the two opposite corners? Ans. 50 rods

3. A ladder 35 feet long reaches the top of a wall 28 feet high. How far is the foot of the ladder from the base of the wall? Ans. 21 feet

Lesson 4

For Step 1, keep in mind Lessons 1, 2, and 3. For Step 2, learn the method of finding the dimensions and area of triangles. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

AREAS OF TRIANGLES

A plane surface has two dimensions, length and breadth. In the different kinds of surfaces which have been described, these two dimensions can be called base and altitude.

The area of a surface is defined as the number of units of surface it contains. It is proportionate to the product of its two dimensions.

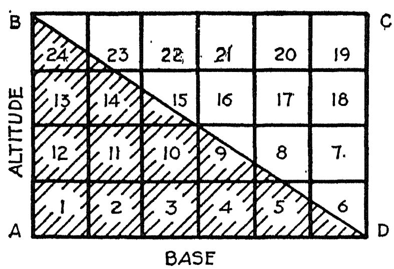


Fig. 29. Graphical Demonstration for the Area of Right Triangle

In the case of the right triangle, we have these two dimensions, as shown in several of the illustrations. In Fig. 29 are shown two right triangles together, so that they form a rectangle. The combined area of these two right triangles is equal, of course, to the area of the rectangle. A unit has been found which equally divides

each side and each end of the rectangle. Thus, by inspection, we can see that the number of square units in this rectangle is 24. This area is found to equal the product when the number of units on one end is multiplied by the number of units on one side, that is, the product when 6 is multiplied by 4. The area, then, is equal to the base times the altitude.

Since the two right triangles are equal in area, the area of each one of them is equal to one-half the base times the altitude.

This statement for the area of a right triangle holds true for all shapes of triangles, so we can state the general rule for finding the area of any triangle.

Rule. *The area of a triangle is equal to one-half the product of the base and altitude.*

This rule can be put into formula form thus:

$$(4)\quad A=\tfrac{1}{2}\,(b\times h) \quad \text{or} \quad \frac{b\times h}{2}.$$

where A represents the area, b the base, and h the altitude of the triangle. Whenever the value of any two of these is given, the value of the other can be found by substituting the given values in the formula.

ILLUSTRATIVE EXAMPLES

1. Find the area of a triangle whose base is 10 inches and whose altitude is 8 inches.

Solution

Instruction	*Operation*
Step 1	**Step 1**
The formula is $A=\frac{b\times h}{}$	
Substitute in this formula the given values for b and h	$A=\dfrac{10\times 8}{2}$
Step 2	**Step 2**
Perform the required operations	$A=\dfrac{10\times \overset{4}{\cancel{8}}}{\cancel{2}}=40$
Area $=40$ sq. in. Ans.	

2. Find the base of a triangle whose area is 100 square feet and whose altitude is 8 feet.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Write the formula	$A=\frac{b\times h}{2}$
Substitute the given values for A and h	$100=\frac{b\times 8}{2}$
Step 2	**Step 2**
Simplify the right side of the formula	$100=b\times\overset{4}{\cancel{8}}$
	$100=b\times 4$
Step 3	**Step 3**
Find the value of b If 4 times b is 100, b equals 100 divided by 4	$100\div 4=25$

Base of triangle is 25 ft. Ans.

3. What is the area of a right triangle whose hypotenuse is 40 feet and whose base is 15 feet?

Solution

Instruction	*Operation*
Step 1	**Step 1**
You will notice that here we are given the value of only one of the letters of the formula. However, we can find the value of h by formula (2), discussed in Lesson 3, $a^2=H^2-b^2$, since h is the third side of the triangle. Substitute the given values in this formula	$a^2=H^2-b^2$ $a^2=40^2-15^2$

Step 2
Perform the indicated operations

Step 2

$$a^2=1600-225$$
$$a^2=1375$$

Step 3
If a^2 is 1375, a is the square root of 1375

Step 3

$$a=\sqrt{1375}=37.08$$

Step 4
Now we have found that the altitude of the triangle is 37.08. To find the area, use formula (4) $A=\dfrac{b\times h}{2}$. Substitute the values for b and h

Step 4

$$A=\frac{15\times 37.08}{}$$

Step 5
Perform the indicated operations

Step 5

$$A=\frac{15\times \cancel{37.08}^{\,18.54}}{}$$

$$A=15\times 18.54=278.10$$

Area is 278.1 sq. ft. Ans.

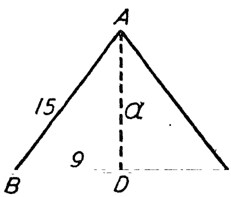
4. Find the area of an isosceles triangle, the base of which is 18 inches and each of the equal sides 15 inches.

Solution

Instruction | *Operation*

Step 1
Here again, as in Example 3, we are given the value of only one letter of the formula $A=\dfrac{b\times h}{2}$. How are we to find the value of h? We must use this fact: When a perpendicular line is drawn from the vertex to the base of an isosceles or equilateral triangle, that perpendicular cuts the base

Step 1



into two equal parts. So, in the accompanying figure, BD is one-half of BC. But we know that BC is 18, therefore BD is 9.

Step 2 Now, consider the right triangle ABD and use formula (2) $a^2 = H^2 - b^2$ to find length of AD. Substitute the given values

Step 2

$$a^2 = 15^2 - 9^2$$

Step 3 Simplify the right side of the equation

Step 3

$$a^2 = 225 - 81$$
$$a^2 = 144$$

Step 4 Take the square root of both sides of the equation

Step 4

$$a = 12$$

Now, AD, or a, is the altitude of the triangle ABC.

Step 5 Use formula (4) to find required area of triangle ABC. b is 18 and a is 12

Substitute the known values in the formula

Step 5

$$A = \frac{b \times h}{2}$$

$$18 \times 12$$

Step 6 Simplify the right side of the equation

Step 6

$$A = \frac{18 \times 12}{2} = 108$$

Area of triangle ABC is 108 sq. in. Ans.

5. Find the area of an obtuse-angled triangle ABC, the base of which is 12 feet and the longest side is 20 feet. The part added to the base is 4 feet.

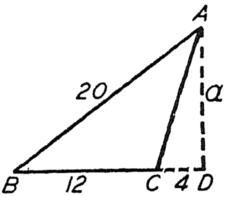
Solution

Instruction | *Operation*

Step 1

Draw a sketch according to the conditions of the problem. The altitude of the triangle ABC is the line AD. (Refer to Fig. 23.) How shall we find the length of this line?

Step 1



Step 2

The triangle ABD is a right triangle of which 20 is the hypotenuse, (12+4) or 16 is one side, and AD is the other side. To find the length of AD, use formula (2) $a^2 = H^2 - b^2$. H is 20 and b is 16. Substitute these values in the formula

Step 2

$$a^2 = 20^2 - 16^2$$
$$a^2 = 400 - 256 = 144$$
$$a = \sqrt{144} = 12$$

Step 3

We have found that the altitude of triangle ABC is 12 feet. To find its area, apply formula (4) $A = \frac{b \times h}{\cdot}$ Substitute the values of b and h in this formula

Step 3

$$A = \frac{12 \times 12}{\;}$$
$$A = 72$$

Area of triangle ABC is 72 sq. ft. Ans.

PRACTICE PROBLEMS

1. An iron brace used in supporting a shelf is fastened to the wall 18 inches below the shelf and to the shelf 12 inches from the wall. Find the length of the brace. Ans. 21.63+ inches

2. The area of a triangle is 24 square inches. If the altitude is 6 inches, find the length of the base. Ans. 8 inches

3. An iron chimney is supported by a guy wire which makes an angle of 63° 24′ with the ground. Determine the angle between the chimney and the wire. Ans. 26° 36′

4. Find the altitude of a triangle whose area is 48 acres and whose base is 48 rods. Ans. 1 mile

5. If one end of a ladder 50 feet long rests on the ground 10 feet from the base of a wall, how far from the ground does the top of the ladder touch the wall? Ans. 49− ft.

6. Find the area of a piece of metal in the shape of an equilateral triangle, each side of which measures 14 inches.

Ans. 84.84+ sq. in.

Lesson 5

For Step 1, keep in mind Lessons 1 and 4. For Step 2, learn the method of finding the dimensions and area of quadrilaterals. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

AREAS OF QUADRILATERALS

In the discussion on the areas of triangles, the method of finding the area of a rectangle was shown in connection with Fig. 29. It is put in rule form as follows:

Rule. *The area of a rectangle is equal to the product of the altitude by the base or the product of any two adjacent sides.*

In Fig. 29, AB is adjacent to BC and also to AD.

The same rule of course holds true for a square. This rule can be put in formula form as follows:

$$(5)\quad A=b\times h$$

where A represents area, b represents base, and h represents altitude.

Since the square is equilateral, the base and altitude are equal, so the formula can be simplified to the following, for a square:

$$(6)\quad A=b^2$$

or

$$A=h^2$$

In the parallelogram, Fig. 13, a rectangle is formed by drawing lines EA and FD perpendicular to the base AD; these lines form right angles at the four corners. The right triangles AEB and DFC are equal. (This is proved in detail in Plane Geometry.)

By formula (5), the area of the rectangle $AEFD$ is the product of base AD and altitude AE. If we subtract the right triangle AEB from this rectangle and add it to the rectangle at the other end, as right triangle FCD, we have the rectangle changed to the parallelogram $ABCD$, and the area of this parallelogram is equal to the area of the rectangle; then formula (6) holds true for parallelograms. You will observe that it will be necessary to find the value of EA or FD before we can proceed to find the area.

Problems relating to parallelograms are so similar to those relating to triangles that special examples are not given.

The trapezoid, Fig. 14, not being a regular figure, cannot have its area calculated by the formulas we have already discussed. In Fig. 14 the dotted diagonal AC is drawn cutting the figure into two triangles ABC and ADC. The sum of the area of these two triangles is the area of the trapezoid. Since the bases are parallel, the altitude of each triangle is the same as that of the trapezoid. Therefore, using formula (4) for a triangle, the areas of these two triangles are: $\frac{1}{2}$ base (BC) $\times$ altitude and $\frac{1}{2}$ base (AD) $\times$ altitude.

$$\text{Sum of these two areas} = \text{(base } (BC) + \text{base } (AD)\text{)} \times \text{altitude}$$

Therefore, put in rule form, we have:

Rule. *The area of a trapezoid is equal to the product of the altitude and one-half the sum of the bases.*

This can be put in formula form also, as follows:

$$(7)\quad A = b+b_1 \times h$$

where b is lower base, b_1 is upper base, and h is altitude.

From this formula the following are obtained:

$$(8)\quad b = \frac{2A}{h} - b_1$$

$$(9)\quad b_1 = \frac{2A}{h} -$$

$$(10)\quad h = \frac{2A}{b+b_1}$$

ILLUSTRATIVE EXAMPLES

1. Find the area of a trapezoid whose bases are 80 rods and 60 rods and whose altitude is 30 rods.

Solution

Instruction	*Operation*
Step 1 Write the formula for finding the area of a trapezoid	**Step 1** $A=\frac{b+b_1}{2}\times h$
Step 2 Substitute the given values in the formula	**Step 2** $A=\frac{80+60}{2}\times 30$
Step 3 Perform the indicated operations Area $=2100$ sq. rd.	**Step 3** $A=\frac{140}{2}\times 30$ $=70\times 30=2100$
Step 4 Reduce this area to acres by dividing by 160 13 acres 20 sq. rd. Ans.	**Step 4** $2100\div 160=13$ and 20 remainder

2. Find the upper base of a trapezoid when the area is 175 square feet, the lower base is 20 feet, and the altitude is 10 feet.

Solution

Instruction	*Operation*
Step 1 Write formula (9) for finding upper base	**Step 1** $b_1=\frac{2A}{h}-b$
Step 2 Substitute the known values for A, b, and h in this formula	**Step 2** $b_1=\frac{2\times 175}{10}-20$

Step 3

Simplify the right side of the equation

Step 3

$$b_1 = \frac{350}{10} - 20$$
$$= 35 - 20 = 15$$

Upper base is 15 ft. Ans.

A **median** is a line drawn parallel to the two bases and midway between them, as MN in Fig. 14.

Rule. *A median to a trapezoid is equal to the sum of the two bases divided by 2.*

3. Find the area of a trapezoid when the median is 45 rods and the altitude is 25 rods.

Solution

Instruction — *Operation*

Step 1

Median $= \frac{1}{2}$ the sum of the bases
First find the sum of the two bases

Step 1

$$45 \times 2 = 90$$

Step 2

Write formula (7) for finding area

Step 2

$$A = \frac{b + b_1}{2} \times h$$

Step 3

We know from Step 1 that $b + b_1 = 90$
Substitute the known values in the formula

Step 3

$$A = \frac{90}{2} \times 25$$

Step 4

Perform the required operations

Step 4

$$A = \frac{90}{2} \times 25$$
$$= 45 \times 25 = 1125$$

Area = 1125 sq. rd. Ans.

NOTE. You will notice that to solve this problem you can multiply the median by the altitude and get the same result as multiplying by 2 and then dividing by 2.

The area of trapeziums as illustrated in Fig. 12 or in Fig. 30 is obtained by dividing the quadrilateral into parts and then finding the sum of the areas of these parts. In Fig. 12 a line BE is drawn parallel to the base AD, and the quadrilateral is thus divided into a triangle and a trapezoid. In Fig. 30 a diagonal is drawn from A to C, and the trapezium has been divided into two triangles.

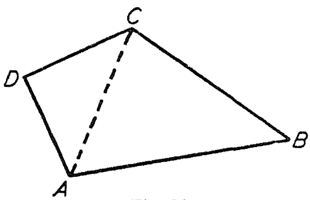
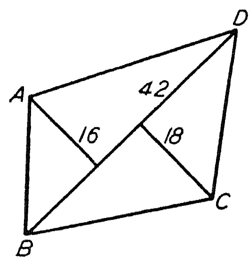


Fig. 30

If certain conditions as to the size of angles and the lengths of sides are given us, we can find the area of such irregular quadrilaterals. However, unless certain relations exist between the angles and the sides, the area of irregular quadrilaterals cannot be found except by the use of instruments or the principles of higher branches of mathematics.

4. Find the area of a trapezium, one of whose diagonals is 42 feet, and the perpendiculars to this diagonal from opposite corners are 16 feet and 18 feet, respectively.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Draw a diagram illustrating the problem	
Step 2	**Step 2**
The diagonal BD has divided the trapezium into two triangles, ABD and BDC. Since we know the length of the base (BD) and the altitudes of these triangles, we can find the areas by formula (4). Write the formula	$A = b \times h$

Step 3
Substituting the given values in the formula

Area of triangle ABD is

Step 3

$$\frac{42\times 16}{2}=336$$

Area of triangle BDC is

$$42\times 18 = 378$$

Step 4
Find total area of the two triangles

Step 4

$$336+378=714$$

Area of trapezium is 714 sq. ft. Ans.

PRACTICE PROBLEMS

1. The area of a rectangle is 10 square feet. Its length is 18 inches. What is its width? Ans. 80 inches
2. A plot of ground with two parallel sides measures 60 feet on one side and 45 feet on the other side and its width is 30 feet. What is its area? Ans. 1575 sq. ft.
3. Find the altitude of a trapezoid whose bases are 30 yards and 20 yards and whose area is 187.5 square yards. Ans. 7.5 yds.
4. The diagonal of a trapezium is 35 feet 6 inches, and the perpendiculars to this diagonal are 9 feet and 3 feet. Find the area of the trapezium. Ans. 213 sq. ft.

Lesson 6

For Step 1, recall the definition and form of a hexagon and of a regular hexagon. For Step 2, learn the relations of the various dimensions of the regular hexagon to one another and the method of finding the area of a regular hexagon. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

AREAS OF REGULAR HEXAGONS

You will recall that a hexagon is a polygon with six sides. In a regular hexagon the six sides are all equal, Fig. 31. To find the area of a surface of this shape, divide it into a number of triangles. This is done by drawing three diagonals BE, AD, and FC. You will observe that there are six triangles in the figure. These triangles

are all equal in area, so that six times the area of any one of these triangles is the area of the hexagon.

Now, draw a line OL perpendicular to the base FE. This line drawn from the intersection of the diagonals and perpendicular

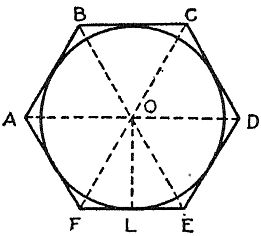


Fig. 31. Regular Hexagon

to the base is called the **apothem.** It is the altitude of the triangle FOE. The base of that triangle is FE.

The area of the triangle FOE by formula (4) is $\frac{1}{2}$ FE times OL. So the area of the six triangles (or the area of the hexagon) is 6 times $\frac{1}{2}$ FE times OL or 3 times FE multiplied by OL. FE is one side of the hexagon and OL is the apothem, so we have the rule for finding the area of a regular hexagon.

Rule. *The area of a regular hexagon is equal to three times the product of the apothem and one side.*

If we let A represent the area and S represent one side of the hexagon and R represent the apothem, we can put the rule into formula form thus:

$$(11) \quad A=3\times S\times R$$

or

$$A=3SR$$

It can be proved by Geometry that OE (as well as the other similar lines) is equal to the side of the hexagon. This indicates that all six triangles are equilateral.

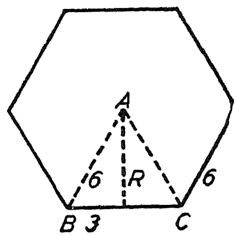
Further, you will recall from Lesson 4 (Illustrative Example 4), that a line drawn perpendicular to the base of an equilateral triangle cuts the base into two equal parts. Hence the line OL cuts the base FE into two equal parts, so that $FL=LE$.

From this it follows that if we know the length of one side of the regular hexagon, we can find its area.

ILLUSTRATIVE EXAMPLES

1. The perimeter of a regular hexagon is 36 feet. What is the length of the apothem?

Solution

Instruction	*Operation*
Step 1 In order to find the apothem, we must first know the length of one side. The sum of the six equal sides is 36 feet, so one side is $36 \div 6 = 6$ feet. Draw an illustrative diagram	**Step 1** 
Step 2 One side BC of the hexagon is the base of the equilateral triangle ABC. The apothem R divides the base into two equal parts. To find the length of R, we use the formula for finding one side of a right triangle. (Lesson 3.)	**Step 2** $R^2 = 6^2 - 3^2$ $R^2 = 36 - 9$ $R^2 = 27$ $R = \sqrt{27} = 5.19+$

Length of apothem = 5.19 ft. Ans.

2. What is the area of a regular hexagon whose sides are 2 inches in length and the length of the apothem is 1.7 inches?

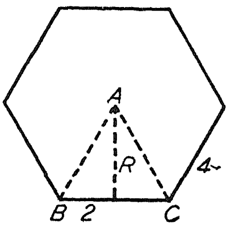
Solution

Instruction	*Operation*
Step 1 Here we can apply directly formula (11) for finding the area of the hexagon. Write the formula	**Step 1** $A = 3 \times S \times R$
Step 2 Substitute the given values in the formula	**Step 2** $A = 3 \times 2 \times 1.7$
Step 3 Perform the multiplications	**Step 3** $3 \times 2 \times 1.7 = 10.2$

Area = 10.2 sq. in. Ans.

3. The perimeter of a regular hexagon is 24 inches. Find its area.

Solution

Instruction	*Operation*
Step 1 In order to find the area, we must know the length of one side and the length of the apothem. If 24 inches is the sum of the six equal sides, then one side is $24 \div 6 = 4$ inches. Draw an illustrative diagram	**Step 1** 
Step 2 Find the apothem R By formula (2) Lesson 3	**Step 2** $R^2 = 4^2 - 2^2$ $R^2 = 16 - 4$ $R^2 = 12$ $R = \sqrt{12} = 3.46+$
Apothem $= 3.46$ in.	
Step 3 Find area of hexagon by formula (11) Substitute the known values of S and R in the formula	**Step 3** $A = 3 \times S \times R$ $A = 3 \times 4 \times 3.46$
Step 4 Perform the multiplications Required area $= 41.52$ sq. in. Ans.	**Step 4** $3 \times 4 \times 3.46 = 41.52$

PRACTICE PROBLEMS

1. The perpendicular drawn from the center to the base of a regular hexagon is 2 feet. The length of one of the sides is 2.31 feet. What is the area of the hexagon? Ans. 13.86 sq. ft.

2. The perimeter of a regular hexagon is 60 inches. What is its area? Ans. 259.8+ sq. in.

Lesson 7

For Step 1, recall the meaning of a "constant." Bear in mind the construction of circles and the meaning of the dimensions by which they are measured. For Step 2, learn how to find the dimensions and areas of circles. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

CIRCLES

A **circle** is a plane figure bounded by a curved line called the **circumference,** every point of which is equally distant from a point within called the **center.** (Fig. 32.)

A **diameter** of a circle is a straight line drawn through the center, terminating at both ends in the circumference.

A **radius** of a circle is a straight line joining the center and the circumference. All radii of the same circle are equal and their length is always one-half that of the diameter.

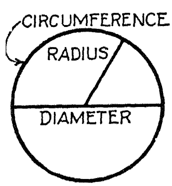


Fig. 32. Circle

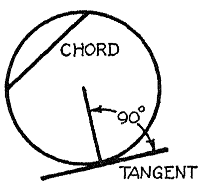


Fig. 33. Chord and Tangent

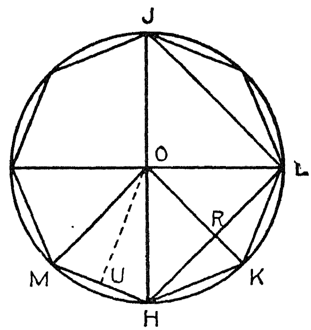


Fig. 34. Octagon with Circumscribed Circle

An **arc** is any part of the circumference of a circle.

An arc equal to one-half the circumference is called a **semi-circumference.**

A **chord** is a straight line joining the extremities of an arc. (Fig. 33.) When a number of chords form the sides of a polygon, the polygon is said to be **inscribed in the circle.** (Fig. 34.)

A circle is said to be **inscribed** in a polygon when its circumference touches all sides of the polygon.

A circle is said to be **circumscribed** about a polygon when the circumference passes through all the vertices of the polygon. Fig. 34.)

A **segment of a circle** is the area included between an arc and a chord. (Fig. 35.)

A **sector** is the area included between an arc and two radii drawn to the extremities of the arc. (Fig. 35.)

A **tangent** is a straight line of unlimited length which touches the circumference at only one point, called the **point of tangency** or **point of contact.** (Fig. 33.) It may be proved by Geometry that a tangent is perpendicular to a radius drawn to the point of tangency.

Concentric circles are circles having the same center. (Fig. 36.)

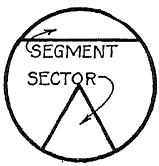


Fig. 35.
Segment and Sector

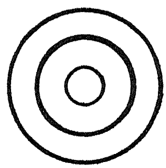


Fig. 36.
Concentric Circles

Relation of Circumference to Diameter. If the lengths of the circumference and the diameter are carefully measured, the ratio, $\frac{\text{circumference}}{\text{diameter}}$, will be found to have a fixed value, whatever size of circle may be selected. This constant ratio has been given the symbol π (which is the Greek letter pi) and has a value of 3.1416 − (approximately $\frac{22}{7}$).

The value $\frac{22}{7}$ for π may be used with safety for rough calculations, but 3.1416 is more nearly correct and should be used when accuracy is desired. Sometimes the value 3.14159 is used.

From the foregoing statements it will be seen that we can readily find the circumference of a circle if the diameter is known.

Rule. *The circumference of a circle is equal to π times the diameter.*

Expressed as a formula

$$(12)\quad C = \pi \times d$$

where C represents the circumference and d the diameter. If r represents the radius of the circle, since the diameter is equal

to twice the radius, we may express the same rule in formula (13) thus,

$$(13)\quad C = \pi \times 2r$$

AREAS OF CIRCLES

The circle may be divided into a number of equal sectors, Fig. 37, that are essentially triangles. The radius of the circle represents the altitude of these triangles and the arc represents the base. Then in the triangle ABO, the area, by formula (4), is $\frac{1}{2}$ ($AB \times$ radius BO). Hence the area of all the sectors, Fig. 38 (or the area of the circle) is one-half the product when the radius is multiplied by the sum of all the arcs. But the sum of all the arcs would be the circumference of the circle. Hence we can state a rule for finding the area of a circle.

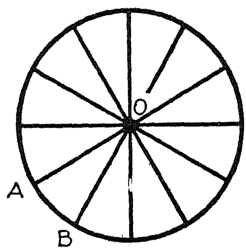


Fig. 37. Circle Divided into Equal Sectors

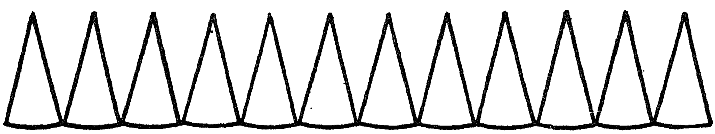


Fig. 38. Sectors of the Circle, Fig. 37, Laid Out

Rule. *The area of a circle is equal to the circumference multiplied by one-half the radius.*

This rule stated as a formula is

$$(14)\quad A = C \times \frac{r}{2}\,.$$

where A represents the area, C the circumference, and r the radius of the circle.

We have already learned by formula (13) that $C = \pi \times 2r$. If we substitute this value for C in formula (14) we get $A = \pi \times 2r \times \frac{r}{2}$.

By cancelling the 2 and performing the multiplication, we arrive at formula

$$(15)\quad A = \pi \times r^2$$

Formula (15) is the one most commonly used for finding the area of any circle. However, conditions in different problems may make one formula more convenient than the other.

ILLUSTRATIVE EXAMPLES

1. Find the circumference of a circle whose radius is 10 feet.

Solution

Instruction	*Operation*
Step 1 Write the formula for finding the circumference when the radius is given. Formula (13)	**Step 1** $C=\pi\times 2r$
Step 2 Substitute the given value for r in the formula, and the value for the constant π	**Step 2** $C=3.1416\times 2\times 10$
Step 3 Perform the indicated multiplications Circumference is 62.832 ft. Ans.	**Step 3** $3.1416\times 2\times 10=62.8320$

2. The circumference of a circular disc is 34 inches. What is its radius?

Solution

Instruction	*Operation*
Step 1 Write the formula involving circumference and radius. Formula (13)	**Step 1** $C=\pi\times 2r$
Step 2 Substitute the values the problem gives	**Step 2** $34=3.1416\times 2r$

Step 3

Perform the operations as indicated

Step 3

$34 = 6.2832 \times r$

Step 4

If 6.2832 times r is 34, r is 34 divided by 6.2832. Perform the division

Step 4

$34 \div 6.2832 = 5.41+$

Radius of disc is 5.41+ in. Ans.

3. What is the area of a circle whose radius is 20 feet?

Solution

Instruction	*Operation*
Step 1 Write the formula for finding area when the radius is given	**Step 1** $A = \pi \times r^2$
Step 2 Substitute the known values in the formula	**Step 2** $A = 3.1416 \times 20^2$
Step 3 Perform the indicated operations	**Step 3** $3.1416 \times 400 = 1256.6400$

Area of circle is 1256.64 sq. ft. Ans.

4. Find the radius of a circular field whose area is 990 square rods.

Solution

Instruction	*Operation*
Step 1 Write the formula that deals with area and radius. Formula (15)	**Step 1** $A = \pi \times r^2$

Step 2
Since the circle is so large, it will be permissible here to use the value $\frac{22}{7}$ for π
Substitute the known values for A and π

Step 2

$$990=\frac{22}{7}\times r^2$$

Step 3
If $\frac{22}{7}$ times r^2 is 990, then r^2 is 990 divided by $\frac{22}{7}$

$r^2=315$

Step 3

$$990\div\frac{22}{7}=990\times\frac{7}{22}$$

$$\overset{45}{\cancel{990}}\times\frac{7}{\cancel{22}}=315$$

Step 4
If the square of r is 315, r will be the square root of 315
Radius of field is 17.7 rd. Ans.

Step 4

$$\sqrt{315}=17.7+$$

5. Find the circumference of a circle whose area is 10 acres.

Solution

Instruction / *Operation*

Step 1
Since length cannot be measured in acres, reduce 10 acres to square rods
10 acres = 1600 sq. rd.

Step 1

$$160\times 10=1600$$

Step 2
We have no formula that deals directly with area and circumference. What formula shall we use? We know that in order to find circumference, we must know either radius or diameter. So use formula (**15**) that deals with area and radius.

Step 2

$$A=\pi r^2$$

Step 3
Substitute the values we know for A and π, using $\frac{22}{7}$ for π since the area is large

Step 3

$$1600=\frac{22}{7}\times r^2$$

Step 4
Find the value of r^2 as in Problem 4, by dividing 1600 by $\frac{22}{7}$

$r^2=509.09$

Step 4

$$r^2=1600\div\frac{22}{7}=1600\times\frac{7}{22}$$

$$=\frac{11200}{22}=509.09+$$

Step 5
To find r (radius), extract the square root of 509.09

Step 5

$$\sqrt{509.09}=22.6\cdot$$

Step 6
Now that we know the radius, we can use formula (**13**) to find the circumference

Step 6

$$C=\pi\times 2r$$

Step 7
Substitute the known values for π and r

Step 7

$$C=\frac{22}{7}\times 2\times 22.6$$

Step 8
Perform the indicated operations

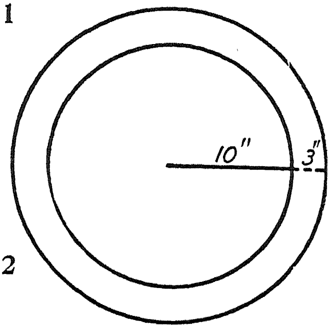
Circumference is 142.1 – rd. Ans.

Step 8

$$\frac{22\times 2\times 22.6}{7}=\frac{994.4}{7}=142.1-$$

5. What is the area of the ring between two concentric circles if the radius of the inner circle is 10 inches and the width of the ring is 3 inches?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Draw a diagram illustrating the circles The area of the ring is the difference between the areas of the two circles	
Step 2	**Step 2**
The radius of the small circle is 10 inches. From the diagram it is evident that the radius of the larger circle is (10+3), or 13 inches. To find the areas use formula (15)	$A = \pi r^2$
Step 3	**Step 3**
Substitute the values for π and r in the formula to find area of large circle, r being 13	$3.1416 \times 13^2 = 3.1416 \times 169$ $= 530.9304$
Area of large circle is 530.9304 sq. in.	
Step 4	**Step 4**
Now, substitute the values for π and r to find area of smaller circle. r is 10	$3.1416 \times 10^2 = 3.1416 \times 100$ $= 314.16$
Area of smaller circle is 314.16 sq. in.	
Step 5	**Step 5**
Find the difference between the two areas	530.9304 314.16 216.7704

Area of ring is 216.7704 sq. in. Ans.

The area of a sector of a circle (Fig. 35) is the same fractional part of the area of the whole circle that the angle of the sector is of the 360° of the circle. Thus if the angle of the sector is 45°, the area of the sector is $\frac{45}{360}$, or $\frac{1}{8}$ of the area of the whole circle.

6. Find the area of a sector of 20° in a circle of 120 feet circumference.

Solution

Instruction	*Operation*
Step 1 Find the radius of the circle by formula (13) Write the formula	**Step 1** $C=\pi\times 2r$
Step 2 Substitute the known values in this formula	**Step 2** $120=\frac{22}{7}\times 2r$ or $120=\frac{44}{7}\times r$
Step 3 If $\frac{44}{7}$ times r is 120, r is $120\div\frac{44}{7}$ Radius is 19.09 ft.	**Step 3** $r=120\div\frac{44}{}$ $=120\times\frac{}{44}=19.09$
Step 4 Find the area of the circle by formula (15) Write the formula	**Step 4** $A=\pi\times r^2$
Step 5 Substitute the known values in the formula	**Step 5** $A=\frac{22}{7}\times(19.09)^2$

Step 6

Perform the indicated operations

Step 6

$\frac{22}{7} \times 19.09 \times 19.09 = 1145.34+$

Step 7

Find the area of the 20° sector

20° is $\frac{20}{360}$ of the entire circle, so

the area of the sector is $\frac{20}{360}$ of the area of the circle

Step 7

$\frac{20}{360}$ of $1145.34 = 63.63$

Area of sector = 63.63 sq. ft. Ans.

Ordinarily the area of a segment cannot be found except by using the principles of Trigonometry; but if sufficient data is given, the area can be arrived at by methods you have already learned.

For example, in Fig. 39 if we know that the angle AOB is 90° and are given the length of the radius, we can find the area of the sector; then find the area of the triangle by formula (4). Subtract the area of the triangle from the area of the sector, and the result is the area of the segment.

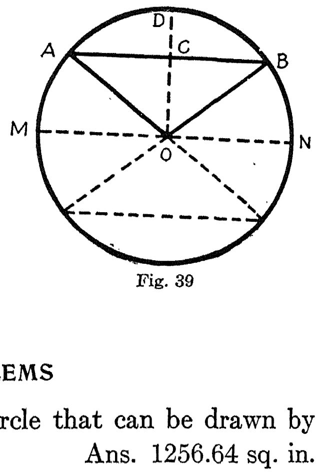


Fig. 39

PRACTICE PROBLEMS

1. Find the area of the largest circle that can be drawn by using as a radius a string 20 inches long. Ans. 1256.64 sq. in.

2. Find the circumference of a wheel that is 7 feet 9 inches in diameter. Ans. 24 ft. 4+ in.

3. The area of a circular lot is 88 square rods. What is its diameter? Ans. 10.58 rd.

4. A circular piece of land whose area is 440 square rods has a roadway around it. The total area of the land and the roadway is 720 square rods. What is the width of the roadway? Ans. 3.3 rd.

5. The diameter of a wheel is 6 feet. How far will it go in making 110 revolutions? Ans. 2073.456 ft.

6. Find the area of the end of a wire whose diameter is .12 inch. Ans. .011309+ sq. in.

FORMULAS

Below is a list of the formulas given in this text. This is for your convenience in making quick references.

Right Triangles

(1) $H^2=a^2+b^2$	H represents hypotenuse
(2) $a^2=H^2-b^2$	a represents one side
(3) $b^2=H^2-a^2$	b represents other side

Triangles

	A represents area
(4) $A=\frac{1}{2}(b\times h)$	b represents base
	h represents altitude

Parallelograms

	A represents area
(5) $A=b\times h$	b represents base
	h represents altitude

Squares

	A represents area
(6) $A=b^2$ or $A=h^2$	b represents base
	h represents altitude

Trapezoids

(7) $A=\frac{b+b_1}{2}\times h$	A represents area
(8) $b=\frac{2A}{h}-b_1$	b represents lower base
(9) $b_1=\frac{2A}{h}-b$	b_1 represents upper base
(10) $h=\frac{2A}{b+b_1}$	h represents altitude

Hexagons

(11) $A = 3S \times R$

A represents area
S represents length of one side
R represents apothem

Circles

(12) $C = \pi \times d$
(13) $C = \pi \times 2r$
(14) $A = C \times \frac{r}{2}$
(15) $A = \pi \times r^2$

C represents circumference
d represents diameter
r represents radius
A represents area
π is a constant value, 3.1416 or $\frac{22}{7}$

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on Page 45.

If you miss more than two of the problems it means you should review the whole Section very carefully.

Do not try this trial examination until you have worked every practice problem in the Section.

Do not start the final examination until you have completed this trial examination.

1. Refer to Fig. 23, page 9. Assume that the dotted line AD is the side of a barn and that line AB is a plank which leans up against the barn so that its top, A, is at the top of the barn. The barn is 16 feet high and the bottom of the plank is 12 feet out from the side of the barn.

(a) What is the length of the plank?

(b) If the barn is made 5 feet higher, how long must the plank be to reach the top without changing the position of the ground end of the plank?

2. One field is square and contains an area of 7,056 square yards. Another is round and has a diameter of 342 feet. If you had to inclose both fields with a 4-wire fence, how many feet of wire would be required?

3. A city lot is 50 feet wide and 120 feet long. A house 30 feet wide and 45 feet long is placed on the lot. The part of the lot not covered by the house is to be seeded for grass. How many square feet will have to be seeded?

4. A square contains an area of 400 square feet. What is the length of one of the diagonals? (The dotted line from B to D, in Fig. 15 page 8, is a diagonal).

5. How many square rods are there in a field whose shape is that of a trapezoid, where the sides are 680 and 450 yards long and where the distance between the parallel sides is 222 feet?

6. A boy has a hoop which has a radius of 12 inches. The boy rolls the hoop at such a speed that it makes 15 revolutions a minute. If the boy rolls the hoop for 30 minutes, how far has he traveled in feet? Use $\pi = 3.1416$.

7. A square table top has sides 5 feet long. What is the area of a round table top having the same perimeter? Use $\pi = 3.1416$.

8. A room measures 18 feet wide and 18 feet long. There is a closet which measures 4 feet wide and 6 feet long. On the floor of the room there is a 12×12-foot rug. The floors are to be varnished everywhere except under the rug. The varnish is extended 12 inches under the rug all the way around. How many square yards of floor will be varnished?

9. A piece of wood used as molding has an end that looks like a right triangle. One side is 6 inches, and the hypotenuse is 12 inches. Find the area of the end in square inches.

10. Assume that the perimeter of a regular hexagon is 48 inches. Assume that diagonals have been drawn from all corners of the hexagon dividing it into 6 triangles. If two of these triangles are removed, what is the area of the remaining part of the hexagon?

FINAL EXAMINATION

1. What is the altitude of a triangle whose base is 360 feet and whose area is 1 acre?

2. How many square yards are there in a rectangle which is 40 feet long and 18 feet wide?

3. A ladder 52 feet long stands flat against the side of a building. How far must it be drawn out at the bottom in order to lower the top 4 feet?

4. What is the radius of a wheel which makes 17,600 revolutions in going 40 miles?

5. A rectangular field is 120 rods long and 40 rods wide. Another field, which is square, has the same perimeter. What is the difference in their areas?

6. A circular plot 24 feet in diameter had a walk 5 feet wide constructed around it. What was the cost of the walk at 35 cents per square yard?

7. How many square feet are there in a board 16 feet long, 18 inches wide at one end and 25 inches wide at the other end?

8. How much farther would a man travel in going around a circular field 100 yards in diameter than in going around a square field whose area is 5625 square yards?

9. The end of an iron rod is in the shape of a regular hexagon. Find the area of the end if its perimeter is 12 inches.

10. The diameter of a circle is 21 feet. What is the area of a sector which makes an angle of 40° at the center of the circle?

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. (a) We know the barn is 16 feet high, so line AD, in Fig. 23, will be 16 feet. The plank is out from the wall, at the ground, a distance of 12 feet, so line DB, in Fig. 23, will be 12 feet.

To find the length of the plank, or line AB, in Fig. 23, we follow **Rule** (a) which is shown at the bottom of page 12. The height of the barn, 16 feet, is one side, and the distance that the ground end of the plank is out from the barn, 12 feet, is another side. We can use **Rule** (a) because triangle ADB is a right triangle.

$$\begin{aligned} 16^2 &= 256 \\ 12^2 &= 144 \\ 256+144 &= 400 \\ \sqrt{400} &= 20 \end{aligned}$$

Note: Review Section 8 if process of finding square root is not clear.

Thus the plank is 20 feet long.

(b) If the barn was built 5 feet higher, then line AD, in Fig. 23, would become $16+5=21$ feet. We would then have the two sides of our triangle 21 feet and 12 feet. Using **Rule** (a), page 12,

$$\begin{aligned} 21^2 &= 441 \\ 12^2 &= 144 \\ 441+144 &= 585 \\ \sqrt{585} &= 24.18 \quad \text{Ans.} \end{aligned}$$

2. The square field has all four sides equal in length. Therefore, we can take the square root of 7,056 and find the length of one side.

$$\begin{array}{r|l} & \overset{\frown}{70}\ \overset{\frown}{56}\ )\underline{84} \\ & 64 \\ 164 & 6\ 56 \\ & 6\ 56 \end{array}$$

Then 84 yards is the length of one side, and all 4 sides are equal in length. Then $84\times4=336$ yards, which is the distance around the square field. Converting yards to feet, $336\times3=1008$ feet, the distance around the field. This is right because 1 yard $=3$ feet. If there are 4 wires in the fence, $4\times1008=4032$ feet of wire required for the fence around the square field.

The round field has a diameter of 342 feet. For a circle the circumference (distance around) is $\pi\times$diameter. (See Formula (12) page 42).

Then $3.1416\times342=1074.4272$. We can call this 1074.43. (You learned how to shorten decimals in Section 5). Thus 1074.43 feet is the distance around the circular field. For 4 wires this would be

$$1074.43\times4=4297.72 \text{ feet}$$

The decimal .72 is over .50 so we can call it 1. Then the wire for the circular field would be 4298 feet.

Adding 4032 and 4298 gives 8330 feet of wire for both fields.

3. The total area of the lot is $120 \times 50 = 6000$ square feet. The area of the house is $30 \times 45 = 1350$ square feet. To find the number of square feet of lot to be seeded, subtract the house area from the lot area.

$$6000 - 1350 = 4650 \text{ square feet. Ans.}$$

4. We have already learned that, where a square is concerned, we can find the length of one side by finding the square root of the area.

$$\sqrt{400} = 20$$

Thus each side is 20 feet long. A diagonal cuts a square into two right-angled triangles. In Fig. 15 the diagonal BD cuts the square $ABCD$ into two right-angled triangles ABD and BCD.

To find the length of the diagonal we can use **Rule (a)**, page 12. The sides of triangle ABD, in Fig. 15, are AB and AD. Both of these sides are 20 feet long.

$$20^2 = 400 \text{ (square of one side)}$$
$$400 + 400 = 800 \text{ (square of two sides added)}$$
$$\sqrt{800} = 28.28 \text{ feet}$$

Then 28.28 feet is the length of the diagonal.

Note: Ordinarily, two decimal places in such an answer are enough.

5. From the description of this field, its shape is like Fig. 14 of page 7. Using this figure as an illustration of the field, the line AD is 680 yards, line BC is 450 yards, and line EF is 222 feet.

To find the area of a trapezoid we use the Rule on page 23.

Before we can find the area, all dimensions must be in the same terms. If we change the 222 feet to yards, then all dimensions will be in yards. Thus $222 \div 3$ (there are 3 feet in one yard) $= 74$ yards.

Now we can follow the Rule. The sum of the bases is $680 + 450 = 1130$ yards. Our rule says we are to use one-half the sum of the bases, so $1130 \div 2 = 565$ yards. Then $565 \times 74 = 41810$ square yards. The 41,810 is *square* yards because it is an area.

From Table V, in Section 7, we know that one square rod equals $30\frac{1}{4}$ square yards. To find the number of square rods in the field, divide 41,810 by $30\frac{1}{4}$.

$$30\tfrac{1}{4} = 30.25 \quad \text{(See Section 5)}$$

```
30.25( 41810.0000 )1382.14
       3025
       11560
        9075
        24850
        24200
          6500
          6050
           4500
           3025
           14750
           12100
            2650
```

Thus the field contains 1382.14 square rods. Here, as usual, we carried the division out so as to give two decimal places in the answer.

6. A hoop is circular like a wheel. Our first calculation will be to find the circumference of the hoop so we will know how far it travels in one revolution.

Formula (13), on page 42, shows that circumference $=\pi\times 2r$. This means $\pi\times 24$ because the radius is 12 inches and $2r=2\times 12=24$. This 24 inches equals 2 feet. Then to find circumference we would have

$$\pi\times 2=3.1416\times 2=6.2832 \text{ feet}$$

Thus the hoop travels 6.2832 feet each revolution.

In one minute the hoop would travel $15\times 6.2832=94.2480$ feet.

If the boy rolled the hoop at this speed for 30 minutes, he would have traveled $94.2480\times 30=2827.4400$ or, roundly, 2827 feet.

7. The square table top has all sides equal in length. Its perimeter is then $5+5+5+5=20$ feet.

If a circular table top has a perimeter of 20 feet the following calculations are necessary to find its area.

Formula (12) on page 42, shows that circumference (the same as perimeter) $=\pi\times d$. If $\pi\times d=$ circumference, then circumference divided by π would give d. Thus,

$$\begin{array}{r}
3.1416\overline{)\,20.0000000\,}6.366 \\
188496 \\
\hline
115040 \\
94248 \\
\hline
207920 \\
188496 \\
\hline
194240 \\
188496 \\
\hline
5744
\end{array}$$

The round table top has a diameter (d) of 6.366 feet.

Now following Formula (15) on page 42, area $=\pi\times r^2$. One-half of 6.366 $=6.366\div 2=3.183$, which is the radius (r). Then $r^2=3.183\times 3.183=10.131489$. We can reduce this to 10.131 as explained in Section 5. Next, $3.1416\times 10.131=31.8275496$. This can be reduced to 31.83 as explained in Section 5. The round table has an area of 31.83 square feet.

8. By studying the statement of this problem we see that if we find the total area of all floors and subtract the area which remains unvarnished under the rug, we will have the number of square yards to be varnished.

The area of the floor of the room is $18\times 18=324$ square feet. The area of the floor of the closet is $4\times 6=24$ square feet. Total floor area is $324+24=348$ square feet.

The problem says that varnishing is to be done 12 inches under the rug all the way around. Thus if we assume the rug is 10 feet by 10 feet we will take care of the foot under the rug. (If you do not understand this assumption of 10×10 feet, draw a diagram of the 12×12 square and mark off 1 foot all around.) Unvarnished area is $10\times 10=100$ square feet.

Subtracting 100 from 348 leaves 248 square feet. There are 9 square feet in a square yard. Then $248\div 9=27\frac{5}{9}$ square yards to be varnished.

9. The end of the molding is a right triangle such as Fig. 21 on page 9. The problem gives the length of one side and the hypotenuse.

We cannot find the area until we know the base and the altitude. (See Rule page 17.)

Refer to Fig. 21 again. Suppose the hypotenuse (AC) is 12 inches and the base (BC) is 6 inches. By the Rule on page 12 we can find the altitude (AB). Then

$$AB^2 = AC^2 - BC^2$$

Substituting actual values

$$AB^2 = 12^2 - 6^2$$
$$AB^2 = 144 - 36$$
$$AB^2 = 108$$
$$AB = \sqrt{108} = 10.39$$

Note: You learned how to find square root in Section 8.

Thus the altitude (AB) is 10.39 inches.

Now we can use Formula (4) for finding the area of a triangle.

$$A = \tfrac{1}{2}(b \times h)$$
$$A = \tfrac{1}{2}(6 \times 10.39)$$
$$A = \tfrac{1}{2}(62.34)$$
$$A = 31.17 \text{ square inches. Ans.}$$

10. To understand this problem, refer to Fig. 31, page 28. We can assume Fig. 31 has a perimeter of 48 inches. Then each side of the hexagon is 8 inches long.

To solve this problem we will find the area of the whole hexagon and then subtract, from that area, the area of the two triangles, such as BOC and COD, which are to be removed. Then we will have the area of the remaining part of the hexagon.

We must find the length of the apothem (R) first. This process is explained on page 29.

$$R^2 = 8^2 - 4^2$$
$$R^2 = 64 - 16$$
$$R^2 = 48$$
$$R = \sqrt{48} = 6.92$$

Length of apothem is 6.92 inches.

Next we use the Rule on page 28. (S represents one side of the hexagon.)

$$A = 3 \times S \times R$$

Substituting

$$A = 3 \times 8 \times 6.92$$
$$A = 166.08$$

Thus the area of the hexagon is 166.08 square inches.

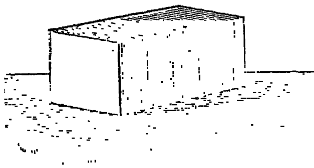
We know that all the 6 triangles in Fig. 31 are equal. Thus each triangle has an area of $\frac{1}{6}$ of $166.08 = 27.68$ square inches. This is approximately correct.

Then two of the triangles have a combined area of 2×27.68 or 55.36 square inches.

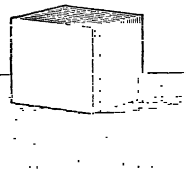
Next subtract 55.36 from 166.08. The result is 110.72 square inches.

Thus the hexagon, minus two of the triangles, has an area of 110.72 square inches.

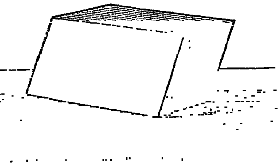
SOLIDS



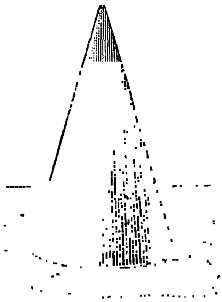
Regular Parallelepiped



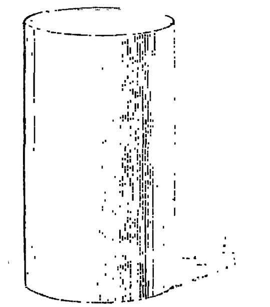
Cube



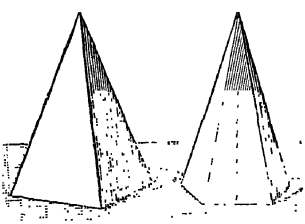
Oblique Parallelepiped



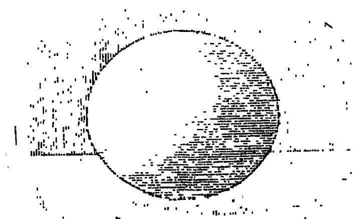
Cone



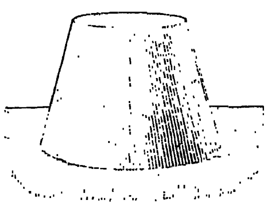
Cylinder



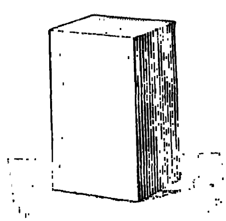
Pyramids



Sphere



Frustum



Prism

PRACTICAL MATHEMATICS

Section 13

MENSURATION—Part II

Lesson 1

For Step 1, keep in mind what you learned in Section 12 relative to areas, because solids are bounded by areas. For Step 2, learn the names and definitions of various solids and the method of finding the areas of prisms. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

SOLIDS

In Section 12, we learned that mensuration is the process of computing the length of lines, the area of surfaces, and the volume of solids. We studied plane surfaces, which are surfaces with only two dimensions. In Section 13, we shall study **solids.** Solids are objects with three dimensions—length, breadth, and thickness. They are of many shapes, the most common of which are prisms, cylinders, pyramids, cones, and spheres. We shall learn how to find the surface area and volume or contents of such objects. It will be necessary to keep the definitions and principles of Section 12 constantly in mind during the study of this text. Frequent reference to them may be necessary.

On the opposite page is shown a group of typical solids. The student should become familiar with these before starting the study of the following text.

PRISMS

A **prism** is a solid whose ends (top and bottom) are equal, similar, and parallel polygons. These ends are called the **bases** of the prism. The sides of the prism are parallelograms. Figs. 1 to 6 are illustrations of prisms.

A prism is named triangular, rectangular, pentagonal, hexagonal, or octagonal according as its bases are triangles, rectangles, pentagons, hexagons, or octagons.

The sides of a prism are called **lateral faces.** It is evident that there will be as many of these lateral faces as there are sides in one of the bases.

The **altitude** of a prism is the perpendicular distance between its two bases. When the bases are perpendicular to the faces, the altitude equals the edge of a lateral face, as in Figs. 1, 3, and 4.

A **right prism** is one whose lateral faces are perpendicular to the bases, as illustrated in Figs. 1, 3, and 4.

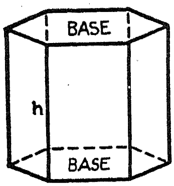


Fig. 1. Right Prism

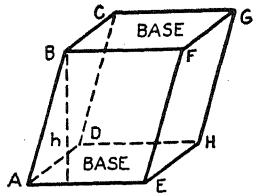


Fig. 2. Parallelopiped

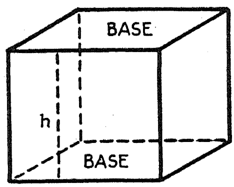


Fig. 3. Rectangular Parallelopiped

An **oblique prism** is one whose lateral faces are not perpendicular to the bases, Fig. 2.

A **parallelopiped** is a prism whose bases are parallelograms, Figs. 2 and 3. If all the edges are perpendicular to the bases, it is called a **right parallelopiped,** Fig. 3.

A **rectangular parallelopiped** is one whose bases and faces are all rectangles, Fig. 3.

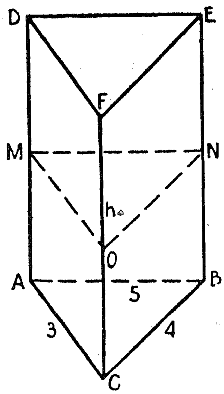


Fig. 4. Right Triangular Prism

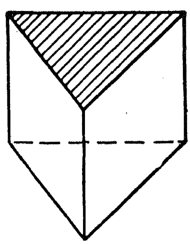


Fig. 5

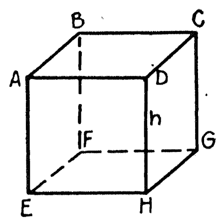


Fig. 6. Cube

A **cube** is a parallelopiped whose bases and faces are all equal squares, Fig. 6.

A **cross section** of a prism is a section that is perpendicular to the edges of the prism. In Fig. 4, the plane MNO is the cross section.

To understand just what a cross section is, imagine that the prism in Fig. 4 had been sawed through following the dotted lines connecting *MNO*. That would cut the prism into two pieces. If the upper piece was moved away, the top of the bottom piece would represent the cross section. This is illustrated in Fig. 5. The shaded area is the cross section.

The **lateral area** of a prism is the combined area of all its faces.

In Fig. 4, the sides *DACF*, *FCBE*, and *EBAD* form the faces. Thus, to find the lateral area of this prism, it is necessary to add the areas of all three faces together.

The **total area** of a prism is the combined area of the lateral faces and the bases.

The bases, in Fig. 4, are *DFE* and *ACB*. Add the areas of these two bases to the areas of all the sides to find total area.

An **edge** is the line where two lateral faces meet, as *FE* or *GH* in Fig. 2.

AREAS OF RIGHT PRISMS

Since the faces or sides of a right prism are all rectangles, the area of one face is found by multiplying its base by its altitude or, in other words, the area of the rectangle is the product of its two adjacent sides. You will recall this fact from Section 12.

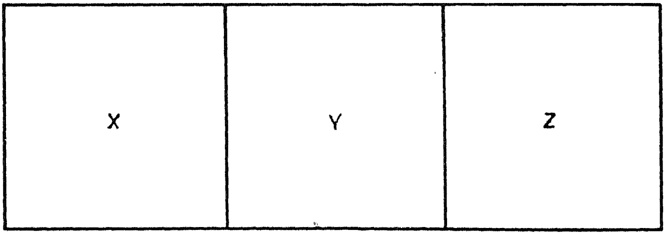


Fig. 7

Refer to Fig. 4. Imagine that the two bases (top and bottom) were taken off. Then imagine that we cut along the side *FC* and moved the side *DACF* back. Do the same for side *FCBE*. This is

called spreading it out. Now look at Fig. 7. The part X represents the side $DACF$, the part Z the side $FCBE$, and Y the side $EBAD$. To close the figure up again, fold on edges DA and EB so that line 1 and 2 could be joined. This would form side FC as in Fig. 4.

It is easily seen that the areas of $X+Y+Z$ would be exactly equal to the areas of the three sides in Fig. 4. Also the distance from 1 to 2 in Fig. 7 is exactly the same as the distance from A to C to B and back to A again, Fig. 4. In other words, the length of Fig. 7 is equal to the perimeter of Fig. 4. The length of line FC in Fig. 4 is the same as DA in Fig. 7. So the width of Fig. 7 is the same as the height or altitude (h) of Fig. 4.

The area of Fig. 7 would equal its length (distance from 1 to 2) times its width (distance DA).

Thus we can state a rule for finding lateral area of a prism. This rule applies for all right prisms such as Figs. 1, 3, 4, and 6.

Rule (1). *The lateral area of a prism equals the perimeter of its base multiplied by its altitude.*

We have seen that the bases of prisms are of different shapes—triangular, rectangular, hexagonal, etc. In Section 12, the methods of finding the areas of such surfaces were discussed. In order to find the area of the bases of a prism, use the method that corresponds to the particular shape of base under consideration.

For example, if a right prism has bases shaped like a triangle (such as Fig. 4) we must find the perimeter of the triangle. If the base happened to be shaped like a hexagon, we would first find the perimeter of the hexagon.

In the case of a cube, the edges are all equal because the faces or sides are all equal. Fig. 6 illustrates this, and you can see that $AB=BC$, $AE=EH$, $HG=GC$, $HD=DA$, etc. Thus we can find the total area of the six faces of a cube as follows:

Rule (2). *The total area of a cube is equal to six times the square of one edge.*

This can be easily understood if we remember that in a cube all six faces are equal. So, to find the area of one face, we multiply length by width. Then, because all six sides are equal, the total area is six times the area of one face. Now, in a square, the length of each side

is the same. So if we square the length of one side we will have the area of that side.

ILLUSTRATIVE EXAMPLES

1. Find the lateral area of a regular pentagonal prism, its altitude being 7 feet and its base measuring 4 feet on each side.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Find the perimeter of the base. Since a regular pentagon has five equal sides, the perimeter will be five times the length of one side	$4 \times 5 = 20$
Step 2	**Step 2**
Multiply perimeter by altitude to find lateral area	$20 \times 7 = 140$
Lateral area $= 140$ sq. ft. Ans.	

2. If the lateral area of a regular pentagonal prism is 140 sq. ft. and the base has sides 4 feet long, what is the altitude?

Note

Before going into the step by step solution we should understand certain things that always hold good as far as areas, rules, etc. are concerned.

We know that in Problem 1 we found the area by multiplying the perimeter by the altitude. Now, if we know the area but do not know the altitude, we can find the altitude by dividing the area by the perimeter. Or, if we want to find the perimeter, when the area and altitude are given, we divide area by the altitude.

We can solve Problem 2, as follows:

Solution

Instruction	*Operation*
Step 1	**Step 1**
Since the given area is the product of the altitude and the perimeter of the base, we can, as explained above, find the altitude by dividing the area by the perimeter. From Problem 1 we know the perimeter is 20 ft.	$140 \div 20 = 7$
Altitude = 7 ft. Ans.	

3. Find the total area of a triangular prism whose base is a right triangle, one of whose sides is 3 inches and the other side 4 inches. The altitude of the prism is 10 inches.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Only two sides of the triangular base are given, so we must first find the third side, or hypotenuse, before we can obtain the perimeter. Apply the formula of the right triangle $H^2 = b^2 + a^2$	$H^2 = 4^2 + 3^2$ $H^2 = 16 + 9 = 25$ $H = \sqrt{25} = 5$
Third side of triangular base is 5 in.	
Step 2	**Step 2**
Add the lengths of the three sides to get the perimeter	$3 + 4 + 5 = 12$
Perimeter is 12 in.	
Step 3	**Step 3**
Multiply the perimeter by the altitude to get lateral area	$12 \times 10 = 120$
Lateral area is 120 sq. in.	

Step 4

Find area of the bases
Apply the formula for finding the area of a triangle $A=\frac{b\times h}{2}$

Step 4

$A=\frac{3\times 4}{2}$

Area of each base is 6 sq. in.
Area of both bases is 12 sq. in.

Step 5

Find total area by adding lateral area and base areas

Step 5

$120+12=132$

Total area is 132 sq. in. Ans.

4. The lateral area of a prism whose base is a regular hexagon is 108 square inches. If the altitude is 9 inches, what is the length of one side of the base?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Since the given area is the product of the altitude and the perimeter of the base and we know the altitude, we can find the perimeter by dividing the lateral area by the altitude	$108\div 9=12$
Perimeter of base is 12 in.	
Step 2	**Step 2**
A regular hexagon has six equal sides, so to find one side divide the perimeter by 6	$12\div 6=2$
Length of one side $=2$ in. Ans.	

PRACTICE PROBLEMS

1. What is the lateral area of a prism whose base is a square having an area of 169 square inches, and whose altitude is 3 feet? Ans. 13 sq. ft.

2. The base of a right prism is in the shape of an equilateral triangle each of whose sides is 6 inches. Find the total area of the prism if its altitude is 15 inches. Ans. 301.17+sq. in.

3. The total area of a prism is 180 square inches. The area of one base is 30 square inches and the altitude is 5 inches. What is the perimeter of the base? Ans. 24 in.

4. What is the total area of a cube one of whose edges measures 6 inches? Ans. 216 sq. in.

Note: Do not go beyond this point in the text until you can solve the above problems and fully understand them.

Lesson 2

For Step 1, keep in mind the shapes of various kinds of prisms and their dimensions. For Step 2, learn how to find the volumes of prisms. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

VOLUMES OF PRISMS

So far in this book, the discussions have been on the surfaces of prisms. Now it is necessary to learn how to find the volumes, or contents, of these solids. The volume of a solid is always given in cubic measure, which you will recall having studied in Section 7.

Volume, or contents, involves the product of three quantities in linear measure or the product of two quantities, one given in square measure and the other in linear measure.

Volume is determined by the number of times the unit of cubic measure (as cubic inch, cubic foot, cubic centimeter, etc.) is contained in the object under consideration. A study of Fig. 8 will illustrate this fact.

In Fig. 8, A represents a cubic inch. It can be seen that there are four cubes similar to A in the bottom layer of B. Also there are three layers in B each containing cubes of the same size as A. Therefore, there are 4×3 or 12 cubes like A in B, or, in other words, there are 12 cubic inches in B.

Each side of the base of B measures 2 inches, so that its area is 2×2 square inches; and since the altitude is 3 inches, the volume is $2\times2\times3$ or 12 cubic inches, as found in the preceding paragraph. The

cubic contents of B, then, can be expressed either as the product of length of base, width of base, and altitude, or as the product of base or end area and altitude.

Rule (3). *The volume of any prism is found by multiplying the area of its base by its altitude.*

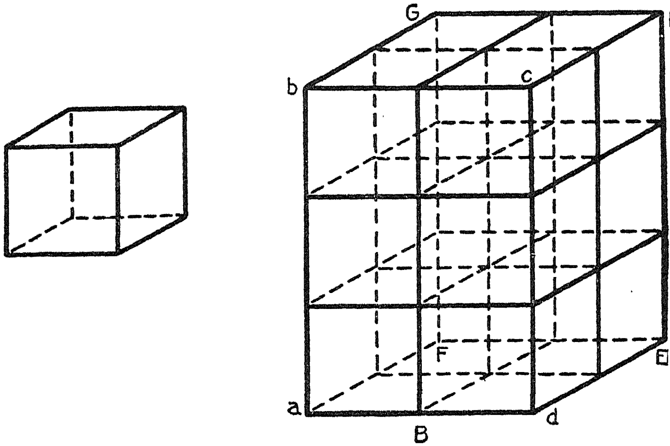


Fig. 8. A—Cubic Inch; B—Parallelopiped

The above rule, as stated, can be used to find the volume of any shape of prism and is therefore the best rule to remember.

The method of finding the area of the base will depend on what the shape of the base is. The methods of finding areas of various shaped bases can be found in Section 12.

The other method of finding the volume of an object is to multiply length × width × breadth. It is all right to use this second method, but it only applies to objects or prisms that are right parallelepipeds such as Figs. 2, 3, 6, and 8. When finding volumes of prisms shaped like Fig. 1 or Fig. 4, we must use **Rule** (3).

It would be advisable to learn to use **Rule** (3) at all times.

There are variations of **Rule** (3) depending on the problem being considered. Sometimes we know the volume of a given prism and also the altitude. In such a case, we could find the base area by dividing the volume by the altitude. Or, if we knew the volume and base area, we could find the altitude by dividing the volume by base area.

The method of finding the area of the base will depend, of course, on the shape of that base, whether it is triangular, rectangular, pentagonal, etc., as we learned in Lesson 1.

ILLUSTRATIVE EXAMPLES

1. What is the volume of a triangular prism whose height is 20 feet? Each end of the prism is an obtuse-angled triangle, the base of which measures 8 feet and the altitude 5 feet.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Find the area of the base or end of the prism by applying the rule for finding the area of a triangle $A=\frac{b\times h}{2}$	
Substituting in the formula	$A=\frac{8\times 5}{2}=20$
Area of base of prism is 20 sq. ft.	
Step 2	**Step 2**
Find the volume of the prism by multiplying the base area by the altitude of the prism	$20\times 20=400$
Volume of prism is 400 cu. ft. Ans.	

2. A certain concrete pillar has a hexagonal cross section. The height of the pillar is 15 feet. Each side of the base measures 2 feet and the line drawn from its center perpendicular to a side is 1.73 feet long. How many cubic yards of concrete are in the pillar?

Solution

Instruction	*Operation*
Step 1	**Step 1**
The base of the prism is a hexagon (as Fig. 1), so to find its area we must apply the rule for the area of a hexagon, which we	

learned in Section 12. $A = 3SR$, where S is one side of the hexagon and R is the apothem. Substituting in this formula

$$A = 3 \times 2 \times 1.73$$
$$= 10.38$$

Base area of prism is 10.38 sq. ft.

Step 2 **Step 2**

Find the volume of the prism by multiplying the base area by the altitude

$$10.38 \times 15 = 155.70$$

Volume of prism is 155.7 cu. ft.

Step 3 **Step 3**

Reduce 155.7 cu. ft. to cubic yards

$$155.7 \div 27 = 5.8-$$

5.8 cu. yd. Ans.

3. The volume of a bar of iron is 1680 cubic inches. It is 4 feet long and its cross-section is in the form of a trapezoid. One of the parallel sides measures 6 inches and the perpendicular distance between the two parallel sides is 5 inches. What is the length of the other parallel side of the end of the bar?

Solution

Instruction *Operation*

Step 1 **Step 1**

Read the problem carefully and draw a diagram to illustrate it, marking the given values in their places

Step 2 **Step 2**

Reduce 4 feet to inches so that all dimensions will be in the same denomination

$$4 \times 12 = 48$$

4 feet = 48 inches

This 48 inches is the altitude of the prism

Step 3

Since the volume of the prism equals the base area multiplied by the altitude, the base area will equal the volume divided by the altitude

$$1680 \div 48 = 35$$

Base area of prism, that is, the area of the trapezoid end is 35 sq. in.

Step 4

We know now the area of the trapezoid and one of its parallel sides or bases. Recall, from Section 12, a formula for finding the second base of a trapezoid when one base and the area are given. It is formula (9)

$$b_1 = \frac{2A}{h} - b$$

Substitute the values we know in this formula

$$b_1 = \frac{2 \times 35}{5} - 6$$
$$= 14 - 6 = 8$$

The other parallel side is 8″. Ans.

PRACTICE PROBLEMS

1. A school room is 40 feet wide and 50 feet long. There are 40 pupils in the room, each requiring 450 cubic feet of air. How high will the room have to be? Ans. 9 ft.

2. Across the bottom, a trough measures 24 inches, across the top 28 inches, and it is 18 inches deep. If the trough when full will hold 243.75 gallons, what is its length? Ans. 10 ft.

3. What is the volume of a prism, the base of which is a right triangle whose two shortest sides are, respectively, 4 feet and 6 feet? The height of the prism is 25 feet. Ans. 300 cu. ft.

4. The perimeter of the base of a regular hexagonal prism is 60 feet. Its height is 20 feet. Find its volume. Find the area of one lateral face. Ans. 5196 cu. ft.; 200 sq. ft.

Lesson 3

For Step 1, recall the method of finding the circumference and area of a circle. For Step 2, acquaint yourself with the form of a cylinder and the method of finding its area and volume. For Step 3, study the Illustrative Examples. For Step 4, work the Practice Problems.

CYLINDERS

A cylinder is illustrated in Fig. 9. Other illustrations are ordinary pipes, such as are used to carry water or gas, a common round lead pencil, and even a broom handle. A cylinder may have any length (h in Fig. 9) and any radius (AO in Fig. 9).

To help illustrate how to find the area of a cylinder, it will be necessary to perform an experiment. On an ordinary piece of paper, draw a rectangle 4 inches long and 2 inches wide. Then cut out this rectangle so you have a piece of paper 2 by 4 inches in size. Letter the corners so that B is in the upper left-hand corner, C in the upper right-hand corner, A in lower left, and D in lower right. Now bend or roll the paper so that CD and BA meet. They should meet so that they form one line, such as BA in Fig. 9. The resulting figure is a cylinder.

A **right cylinder** is one whose side is perpendicular to its base. This means that BA, in Fig. 9, is perpendicular to AO.
A **circular cylinder** is one whose bases are circles.
This means that both the top and bottom, in Fig. 9, are circles.
Only right circular cylinders will be considered in this text.

AREAS OF CYLINDERS

Rule (4). *The lateral area of a cylinder is found by multiplying the circumference of the base by the altitude.*

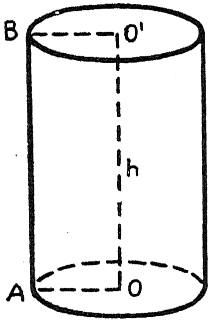


Fig. 9. Cylinder

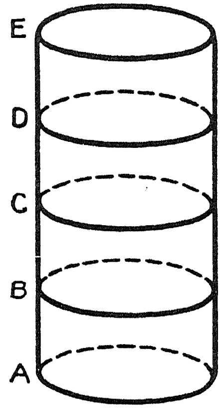


Fig. 10. Cylinder Divided into Equal Sections

Rule (4) can be explained by making use of the experiment you made with the piece of paper 3×2 inches in size. The paper is in the form of a rectangle and you know the area of a rectangle is found by multiplying the length by width. Thus $3 \times 2 = 6$ which is the area of the rectangle. Now, after the paper was rolled so as to form a cylinder, its area was not changed but its appearance was changed. The length after being rolled becomes the circumference of the cylinder. The vertical dimension of the original paper (2″) becomes the height of the cylinder or its altitude. So if we multiply the circumference (3″) by the altitude (2″) we get 6 sq. inches which is the same area as the original piece of paper.

There are variations of **Rule** (4) as there were to **Rule** (3). If we know the lateral area and the circumference, we can find the altitude by dividing the lateral area by the circumference.

Also, if we know the lateral area and the altitude, we can find the circumference by dividing the lateral area by the altitude.

The total area of a cylinder, as in the case of a prism, is the sum of the lateral area and the areas of the two bases. The area of each base, for all purposes of this text, is the area of a circle and is found by the methods learned in Section 12.

VOLUMES OF CYLINDERS

In Fig. 10 a cylinder is divided into four equal sections, each one unit high. When the area of one circular base is multiplied by one unit of height, the volume of that section is obtained; so when the area of the base is multiplied by the four units of height, the volume of the four sections or the volume of the cylinder is obtained.

Hence we can state the rule for finding the volume of a cylinder:

Rule (5). *The volume of a cylinder is found by multiplying the area of its base by its altitude.*

Section 12 shows the method of finding the area of such a base.

ILLUSTRATIVE EXAMPLES

1. Find the number of square feet of sheet metal necessary to cover the sides and bottom of a cylindrical tank 15 feet long and 7 feet in diameter.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Find the area of the bottom, which, of course, is a circle. Section 12 gives the formula for doing this: $A = \pi r^2$. Find the radius (r) by dividing the diameter by 2	$7 \div 2 = 3.5$
Substitute the given values in the formula	$A = \frac{22}{7} \times (3.5)^2$ $= \frac{22}{7} \times 12.25 = 38.5$
Area of the bottom is 38.5 sq. ft.	
Step 2	**Step 2**
Find the lateral area. To do this we must first find the circumference of the base	
Formula is $C = \pi d$	

Substitute the known values in this formula

$$C=\frac{22}{7}\times 7=22$$

Circumference is 22 ft.

Step 3

Find the lateral area by multiplying the circumference by the altitude

Step 3

$$22\times 15=330$$

Step 4

Find the required area of sides and bottom by adding the two areas we have found

368.5 sq. ft. Ans.

Step 4

$$330+38.5=368.5$$

2. The lateral area of a cylinder is 981.75 square feet. If its altitude is 50 feet, find the radius of its base.

Solution

Instruction	*Operation*
Step 1 Since the lateral area is the product of the altitude and circumference, we can find the circumference by dividing the lateral area py the altitude Circumference = 19.635 ft.	**Step 1** $981.75\div 50=19.635$
Step 2 The formula for finding the circumference is $C=2\times \pi\times r$. Substitute the known values in this formula	**Step 2** $19.635=2\times\frac{22}{7}\times r$

Step 3
To clear of fractions, multiply both sides of the equation by the denominator of the fraction

Step 3

$$19.635 \times 7 = 2 \times \frac{22}{7} \times r \times 7$$
$$137.445 = 44 \times r$$

Step 4
If 44 times r is 137.445, r equals 137.445 divided by 44
Required radius is 3.124 ft. Ans.

Step 4

$$137.445 \div 44 = 3.124-$$

3. When blasting in rock, a hole 95 feet long and 10 feet in diameter was made. How many cubic yards of rock were removed?

Solution

Instruction — *Operation*

Step 1
The hole will be in the form of a cylinder, so to find its cubic contents or volume, we must first find the area of the end. Our formula is $A = \pi r^2$. r is $\frac{1}{2}$ of 10, or 5 feet.
Substitute the values in the formula

Step 1

$$A = \frac{22}{7} \times 5^2$$
$$= \frac{22}{7} \times 25 = 78.57$$

Base area is 78.57 sq. ft.

Step 2
Find the volume by multiplying base area by the length (i.e., the altitude of the cylinder)
Volume is 7464.15 cu. ft.

Step 2

$$78.57 \times 95 = 7464.15$$

Step 3

Reduce this to cubic yards by dividing by 27

276.45 cu. yd. Ans.

Step 3

$7464.15 \div 27 = 276.45$

4. A contractor removed 10,000 cubic yards of clay from the location for a drain. The drain is 6 feet in diameter. How long is it?

Solution

Instruction	*Operation*
Step 1	**Step 1**
We know that the volume of the cylindrical-shaped drain is the product of the area of the end and the length, so if we find the area of one end we can find the length. Radius is 3 ft. $A = \pi r^2$ Substitute the given values in the formula	$A = \frac{22}{7} \times 3^2$ $= \frac{22}{7} \times 9 = 28.28+$
Area of end is 28.28 sq. ft.	
Step 2	**Step 2**
Find the length by dividing the volume by this area. Since the volume is given in cubic yards, it must be reduced to cubic feet. Divide this volume by area of end	$10{,}000 \times 27 = 270{,}000$ $270{,}000 \div 28.28 = 9547.3+$

Length of drain is 9547.3+ft. Ans.

PRACTICE PROBLEMS

1. What must be the diameter of a tank with a circular base if it contains 2000 cubic feet and is 20 feet high? Ans. 11.28 ft.

2. A cistern is 6 feet in diameter and 8 feet deep. How many gallons of water will it hold? Ans. 1697.1 gals.

3. Find the number of square inches of material necessary to construct a box (including the lid) if the base is circular and the material is used two-ply thick in making the base. The box is 5 inches high and has a diameter of 3 inches. Ans. 68.32+ sq. in.

4. Assume a cylindrical shaped post measuring 72 inches long and having a radius of 4 inches. If a hole 3 inches in diameter was bored vertically through the center of the post, what would be the volume of the part remaining? Give answer in cubic inches.

Ans. = 3110.1840 cu. inches

Lesson 4

For Step 1, recall what you learned in Book No. 12 about triangles, because the faces of a pyramid are triangles. For Step 2, study the construction of pyramids and the rules for finding their area and volume. For Step 3, work the Illustrative Examples. For Step 4, solve the Practice Problems.

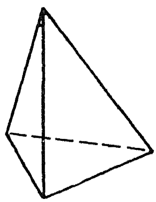


Fig. 11. Triangular Pyramid

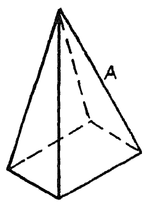


Fig. 12. Quadrangular Pyramid

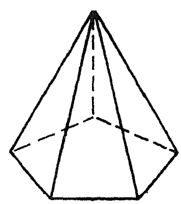


Fig. 13. Pentagonal Pyramid

PYRAMIDS

A **pyramid** is a solid whose base is a polygon and whose sides are triangles. The triangles meet in a common point to form the **vertex** of the pyramid.

This means that any solid figure whose base is a polygon and whose sides are triangles such as *ADB*, in Fig. 14, or *DOC*, in Fig. 15, is a pyramid. The triangles must all meet at one point, called the vertex, such as *D* in Fig. 14.

The **altitude** of the pyramid is the perpendicular distance from the vertex to the base, such as line *OE* in Fig. 15.

Pyramids are named according to the kind of polygon forming the base, namely, triangular, Fig. 11, quadrangular, Fig. 12, pentagonal, Fig. 13, hexagonal, Fig. 14.

A **regular pyramid** is one whose base is a regular polygon and whose vertex lies in a perpendicular erected at the center of the base, Figs. 13, 14, and 15.

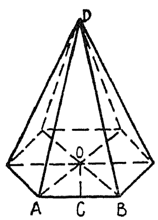


Fig. 14. Hexagonal Pyramid

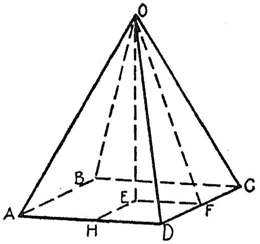


Fig. 15. Pyramid Showing Altitude and Slant Height

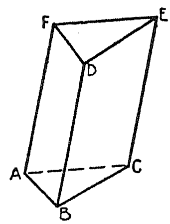


Fig. 16. Triangular Prism

This means that if a line from the center of the base and at right angles to the base is erected, the vertex of the pyramid must be on this perpendicular line in order to call the pyramid *regular*.

The **slant height** of a regular pyramid is a line drawn on a side from the vertex and perpendicular to a side line of the base. (See the line *OF*, Fig. 15.) In other words, it is the altitude of one of the triangles which form the sides.

The **lateral edges** of a pyramid are the intersections of the triangular sides.

The triangles forming the sides of the pyramid are called the **faces.**

AREAS OF PYRAMIDS

The lateral area is the combined area of all the triangles forming the sides. Now, the base of each of these triangles is one side of the base of the pyramid and the altitude of each of these triangles is the slant height of the pyramid, since the slant height is perpendicular to the base. (Distinguish carefully between the altitude of the pyramid

itself and the altitude of one of its faces.) In Fig. 15, the altitude of the pyramid itself is shown by the dotted line EO, while the altitude of one of the sides or faces is shown by the dotted line OF.

Rule (6). *The lateral area of a pyramid is equal to the perimeter of the base multiplied by one-half the slant height.*

This rule seems right when we remember that the area of a triangle is the base times ½ the altitude. In a pyramid where there are several triangles, we add all the bases and multiply this sum by ½ the slant height.

If the slant height is not given, we can easily find it by the law of the right triangle. For example, suppose in Fig. 15 we know the length of each side of the base and the length of the altitude or line OE. Then, line EF is equal to one-half line AD because E is the center point of the bases. Thus, knowing lengths of EF and EO is the right triangle of OEF, we can easily find the hypotenuse or slant height of OF.

Keep in mind that lateral area means only the area of all the sides or faces and does not include the area of the base.

The total area of a pyramid is equal to the sum of the lateral area and the area of the base. The method of finding the area of the base will be determined by the shape of the base.

VOLUMES OF PYRAMIDS

A study of Fig. 16 will show that a triangular prism may be divided into three equal pyramids—$DABC$, $CEFD$, and $ACDF$.

We have already learned, **Rule (3)**, that the volume of a prism is found by multiplying the area of the base by the altitude. Fig. 16 is a prism so this would be true of it. In Fig. 17, we see that the prism is divided into three separate pyramids. Each of the pyramids in Fig. 17 is one-third of Fig. 16. So, the following rule will be clear.

Rule (7). *The volume of a pyramid is one-third its base area times its altitude.*

It should be remembered that there are variations of **Rule (7).** For example, if we knew the volume and the altitude, we could find one-third of base area by dividing the volume by the altitude.

The student should study over these rules and variations until he feels sure he thoroughly understands them in order to prevent confusion at a later time in his studies.

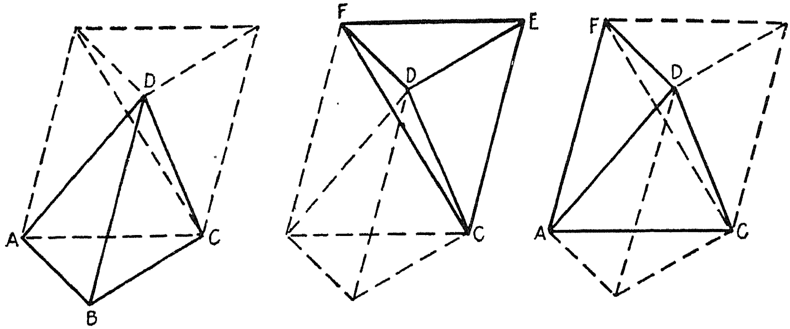


Fig. 17. Showing the three pyramids that make up the prism shown in Fig. 16.

ILLUSTRATIVE EXAMPLES

1. Find the lateral area of the pyramid shown in Fig. 15. Its altitude OE is 12 feet and each side of the square base is 8 feet.

Solution

Instruction	*Operation*
Step 1	**Step 1**
The rule for finding the lateral area involves the slant height, so our first step is to find the slant height. From the center of the square base to the center of one side is half one side. So the line EF is 4 feet. The line OE is 12 feet and OEF is a right triangle, of which the slant height OF is the hypotenuse.	
Apply the law of the right triangle	$(OF)^2=4^2+12^2$ $(OF)^2=16+144=160$ $OF=\sqrt{160}=12.65-$
Slant height $=12.65$ ft.	

Step 2	**Step 2**
Find the lateral area by applying Rule (6). Since the base of the pyramid is a square, 8 feet on each side, its perimeter is 32 ft.	Area $= 32 \times \frac{1}{2}$ of 12.65 $= 202.4$
Lateral area is 202.4 sq. ft. Ans.	

2. The volume of a pyramid with a square base is 135 cubic feet. Its altitude is 15 feet. Find the perimeter of its base.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Since the volume of this pyramid (135 cu. ft.) is one-third of the product of the altitude and the base area, **Rule (7)**, then the product of the base area and the altitude is three times 135	$135 \times 3 = 405$
Step 2	**Step 2**
Since the base area multiplied by the altitude is 405 cu. ft. and the altitude is 15 feet, the base area will equal 405 divided by 15. Base area is 27 sq. ft.	$405 \div 15 = 27$
Step 3	**Step 3**
The base is a square, so we can find the length of one side by taking the square root of the area. One side of the base of the pyramid is 5.196 ft.	$\sqrt{27} = 5.196$
Step 4	**Step 4**
The perimeter of the square base will be four times the length of one side	$5.196 \times 4 = 20.784$

Perimeter of base of pyramid is
20.784 ft. Ans.

3. Find the volume of a regular hexagonal pyramid. One side of the base measures 6 feet and the altitude of the pyramid is 12 feet.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Refer to Fig. 14. Before we can find the volume we must know the area of the hexagon that forms the base of the pyramid. Recall what you learned about hexagons in Book No. 12. The triangle AOB is equilateral, each side being 6 feet. The apothem OC bisects the base, so that CB will measure 3 feet, and OCB is a right triangle.	
Apply law of right triangle to find apothem OC	$(OC)^2=(OB)^2-(CB)^2$ $(OC)^2=6^2-3^2$ $(OC)^2=36-9=27$ $OC=\sqrt{27}$ $OC=5.196$
OC (apothem) $=5.196$	
Step 2	**Step 2**
The area of a hexagon is three times one side multiplied by the apothem. Three times one side is 18 feet	$18\times5.196=93.528$
Area of base of pyramid is 93.528	
Step 3	**Step 3**
Find the volume of the pyramid by taking one-third of the	

product of the base area and the altitude $\frac{1}{3} \times 93.528 \times 12 = 374.112$

Volume is 374.112 cu. ft. Ans.

PRACTICE PROBLEMS

1. Find the total area of a pyramid with a square base measuring 3 feet on a side if the slant height is 9 feet. Ans. 63 sq. ft.
2. Find the volume of a pyramid whose base is an equilateral triangle 12 feet on each side and whose altitude is 50 feet. Ans. 1039+cu. ft.

Lesson 5

For Step 1, bear in mind what you have learned about circles. For Step 2, note the form of a cone and learn how to find its area and volume. For Step 3, study the Illustrative Examples. For Step 4, solve the Practice Problem.

CONES

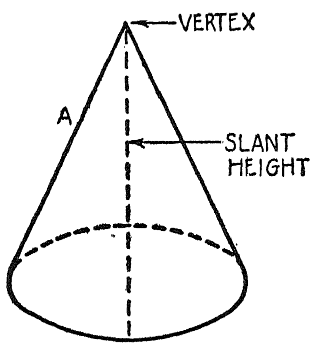


Fig. 18. Circular Cone

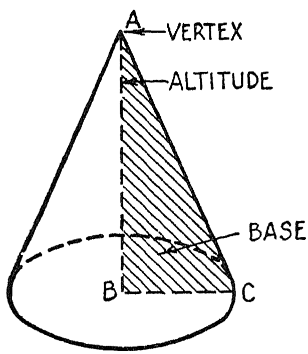


Fig. 19. Cone of Revolution

A cone is a solid whose base is a circle and whose surface tapers from the base to a point called the vertex or top, Figs. 18 and 19. A cone may be considered a pyramid with so many sides or faces that each face would be so small as to make actual counting impossible and so small that the surface or lateral area would appear smooth.

The **altitude** of a cone is the perpendicular distance from the vertex to the base.

The **slant height** is the distance from the vertex to any point on the circumference of the base.

The **lateral area** of a cone is the area of the tapering side.

AREAS AND VOLUMES OF CONES

The rules for finding areas and volumes of cones are the same as the rules for finding areas and volumes of pyramids. The reason for this, as already explained, is because a cone may be considered a pyramid with an unlimited number of sides.

Rule (8). *The lateral area of a cone is found by multiplying the circumference of the base by one-half the slant height.*

Rule (9). *The volume of a cone is one-third of the product of its base area and altitude.*

ILLUSTRATIVE EXAMPLES

1. A conical steeple is 100 feet high and the base is 25 feet in diameter. Find the cost of painting its lateral surface at 40 cents per square yard.

Solution

Instruction	*Operation*
Step 1	**Step 1**
The surface to be painted is the lateral area of the cone. So, first, find the circumference of the base by the formula $C = \pi d$	$C = 3.1416 \times 25$ $= 78.54$
Circumference $= 78.54$ ft.	
Step 2	**Step 2**

Find the slant height, using the law of the right triangle. (Refer to Fig. 19.) One side of the right triangle is the altitude (100 ft.). The other side of the triangle is half of the diameter or 12.5 ft. The hypotenuse is the slant height.

Apply law of right triangle	(Slant height)$^2 = 100^2 + 12.5^2$ $= 10000 + 156.25$ $= 10156.25$ Slant height $= \sqrt{10156.25}$ $= 100.77$
Step 3 Find the lateral area by **Rule** (8). Multiply the circumference of the base by half the slant height	**Step 3** $78.54 \times \frac{100.77}{2} = 3957.2379$
Area to be painted $= 3957.24-$ sq. ft.	
Step 4 Reduce area in square feet to square yards by dividing by 9 Find cost at 40 cents per sq. yd. Required cost is \$175.87. Ans.	**Step 4** $3957.24 \div 9 = 439.69$ $439.69 \times 40 = 17587.60$

2. Both the circumference of the base and the slant height of a cone are 25 inches. Find its volume.

Solution

Instruction	*Operation*
Step 1 Finding volume requires both altitude and the base area. Neither are given, so we must find them. (Refer to Fig. 19.) First, we will find the radius so we can find base area. If $\pi \times d$ = circumference, then d will equal circumference divided by π. This is a variation of the formula for finding circumference. Having found d we know $r = \frac{1}{2}$ of d.	**Step 1** $\pi d =$ Circumference $d =$ Cir. $\div\ \pi$
Radius $= 7.95 \div 2 = 3.98$	$d = 25 \div 3.1416$ $= 7.95$

Step 2
Find area of the base. The base is a circle. From Section 12 we know that area $= \pi r^2$. We found the radius in Step 1.

Step 2

$$\text{Area} = \pi r^2$$
$$= 3.1416 \times (3.98)^2$$
$$= 49.7642$$

Step 3
Now we can find the altitude of the cone by law of right triangle. In Fig. 19, we know the slant height $(AC) = 25$ inches.
This is the hypotenuse of a right triangle ABC. The radius (BC) is 3.98 as we have already found. We wish to find the altitude or line AB. Apply law of right triangle and solve.
Altitude $= 24.6$

Step 3

$$(\text{Altitude})^2 = 25^2 - (3.98)^2$$
$$= 625 - 15.84$$
$$= 609.16$$
$$\text{Altitude} = \sqrt{609.16}$$
$$= 24.6$$

Step 4
Find the volume of the cone by **Rule (9).** It is one-third of the product of base area and altitude

Step 4

$$\tfrac{1}{3} \times 49.7642 \times 24.6 = 408.06+$$

Volume of cone is 408.06 cu. in. Ans.

PRACTICE PROBLEM

1. Find the volume and lateral area of a cone 20 feet high, the radius of the base being 5 feet. Ans. Volume is 523.6 cu. ft.; lateral area is 323.58+sq. ft.

Lesson 6

For Step 1, bear in mind what you have learned about pyramids and cones. For Step 2, learn the meaning of "frustum" and the method of finding its lateral area and volume. For Step 3, work the Illustrative Examples. For Step 4, work the Practice Problems.

FRUSTUMS

Refer to Figs. 12 and 18. Suppose that we could, starting at the point A in each figure, saw or cut off the top making sure that the line on which we cut was parallel to the base in each case. The top portion can be moved away. We would then, after cutting Fig. 12, have a figure such as Fig. 20 and after cutting Fig. 18 we would have left a figure such as Fig. 21. These parts (Figs. 20 and 21) are called frustums.

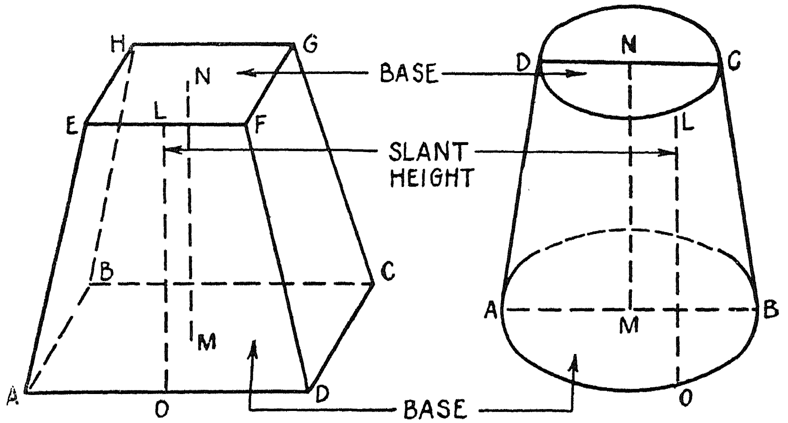


Fig. 20. Frustum of Pyramid Fig. 21. Frustum of Cone

The **altitude** of a frustum is the length of the perpendicular between bases, as lines $M\ N$ of Fig. 20.

The **slant height** of a frustum is the shortest distance between the perimeters of the bases and is shown by lines OL.

LATERAL AREAS OF FRUSTUMS

In Fig. 20, line OL is the altitude of the trapezoid $AEFD$; therefore, the lateral area of the frustum is equal to the sum of the areas of the four trapezoids composing its faces. Note that the altitude of the trapezoid is the slant height of the frustum. Since the area of one trapezoid is equal to one-half the sum of the bases times the altitude (Section 12), the rule for the lateral area of a frustum is:

Rule (10). *The lateral area of the frustum of a right pyramid equals one-half the sum of the perimeters of the two bases times the slant height.*

Since a cone may be considered as a pyramid with sides so numerous and so small that the surface appears smooth, a similar rule will be used for finding the lateral area of a frustum of a cone.

Rule (11). *The lateral area of the frustum of a cone is found by multiplying half the sum of the circumferences of the two bases by the slant height.*

The total area of a frustum is the sum of the lateral area and the two bases.

VOLUMES OF FRUSTUMS

The explanation for the rule for finding the volume of a frustum of a pyramid or cone is too difficult to be introduced here. Only the rule and its application in a problem will be given.

Rule (12). *To find the volume of a frustum take the sum of the areas of the two bases; to this add the square root of the product of the two bases; multiply the result by one-third of the altitude.*

ILLUSTRATIVE EXAMPLES

1. Find the lateral area of the frustum of a cone, the slant height of the frustum being 42 feet and the radii of the bases are 12 feet and 4 feet, respectively.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Apply **Rule (11).** Find the circumference of larger base	$C=2\times 3.1416\times \mathbf{12}$ $=75.3984$
Step 2	**Step 2**
Find the circumference of smaller base	$C=2\times 3.1416\times 4$ $=25.1328$

Step 3 Find half the sum of these circumferences	**Step 3** $75.3984+25.1328=100.5312$ $100.5312 \div 2=50.2656$
Step 4 Find lateral area by multiplying by 42 Lateral area of frustum is 2111.15+ sq. ft. Ans.	**Step 4** $50.2656 \times 42=2111.1552$

2. What is the volume of the frustum of a square pyramid, the sides of whose bases are 2 feet and 8 feet, the altitude of the frustum being 15 feet.

Solution

Instruction	*Operation*
Step 1 Apply **Rule (12).** Find area of the bases, each base being a square	**Step 1** $2\times2=4$ $8\times8=64$
Step 2 Find the square root of the product of the two bases	**Step 2** $\sqrt{4\times64}=\sqrt{256}=16$
Step 3 Find sum of the two bases and this root	**Step 3** $4+64+16=84$
Step 4 Multiply the sum by one-third of altitude	**Step 4** $84\times\frac{1}{3}$ of 15 $84\times5=420$

Volume of frustum $=420$ cu. ft. Ans.

PRACTICE PROBLEMS

1. Find the entire surface area of the frustum of a cone whose slant height is 50 feet and the radii of whose bases are 10 feet and 5 feet. Ans. 2748.9 sq. ft.

2. What is the volume of a frustum of a cone 24 feet in altitude if the radius of its top is 7 feet and the radius of the bottom 14 feet? $\left(\text{Use } \pi=\frac{22}{7}\right)$. Ans. 8624 cu. ft.

Lesson 7

For Step 1, recall what you have learned about circles. For Step 2, note the shape of a sphere and learn the rules for finding its surface area and volume. For Step 3, work the Illustrative Examples. For Step 4, solve all the Practice Problems.

SPHERES

A **sphere** is a solid bounded by a curved surface every point of which is equally distant from a point within called the **center.** In other words, it is a perfectly round ball, Fig. 22.

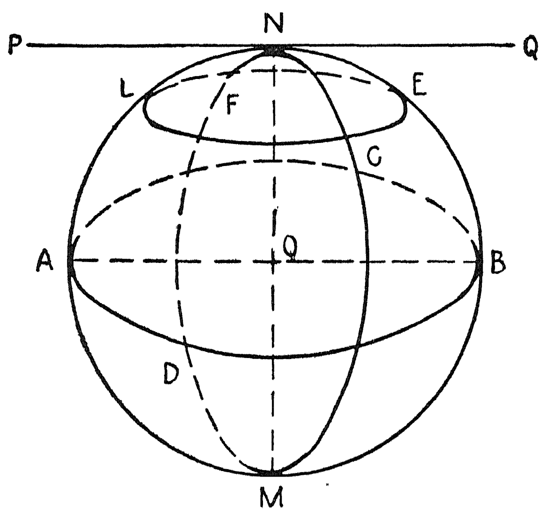


Fig. 22. Sphere

The **diameter** is a straight line drawn through the center and having its extremities in the curved surface, as AB.

The **radius** is a straight line from the center to a point on the surface; it is equal to one-half the diameter. OM, ON, OA, OB are all radii.

A plane is **tangent** to a sphere when it touches the sphere at only one point, as plane PNQ touching at N. The plane PNQ can be thought of as a large sheet of paper touching the circle at N.

When a plane cuts through a sphere, the section is a circle such as plane $ACBD$.

When the plane cuts through the center of the sphere, the resulting section is called a **great circle.** $ANBM$, $ACBD$, and $NCMD$ are great circles.

When the plane does not pass through the center, the section is called a **small circle,** as $LCEF$.

The **circumference** of a sphere is the same as the circumference of a great circle.

AREAS OF SPHERES

It is proven by Geometry that the following statement is true: The area of the surface of a sphere equals four times the area of one of its great circles.

Now we learned in Section 12 that the area of a circle is found by multiplying the square of the radius by π. We can therefore state that the surface of a sphere is four times the square of the radius multiplied by π.

Put in formula, this would be stated $S=4\pi r^2$, where S represents the surface of the sphere and r the radius.

Now r can be expressed as $\frac{d}{2}$; so if we substitute $\frac{d}{2}$ for r in the formula, we get $S=4\pi\left(\frac{d}{2}\right)^2$ or $S=4\pi\frac{d^2}{4}$ which reduces to $S=\pi d^2$. This is the formula most generally used for finding the surface of a sphere and is perhaps the most easily remembered.

Rule (13). *The surface of a sphere is π times the square of the diameter.*

VOLUMES OF SPHERES

The following rule can be proved by Solid Geometry:

Rule (14). *The volume of a sphere equals the area of the surface multiplied by one-third of the radius.*

This can be expressed in formula in this way: $V=\frac{1}{3}Sr$, where V represents the volume, S the surface, and r the radius. But we know that $S=4\pi r^2$, therefore $V=\frac{1}{3}$ of $4\pi r^2 \times r$ (substituting $4\pi r^2$ for S), which simplifies to $V=\frac{4}{3}\pi r^3$. Again, since $r=\frac{d}{2}$, the formula can be put in the form

$$V=\frac{4}{3}\pi\left(\frac{d}{2}\right)^3=\frac{4}{3}\pi\frac{d^3}{8}=\frac{1}{6}\pi d^3$$

Summing up the formulas pertaining to spheres, then, we have

$$S=4\pi r^2 \qquad \text{or } S=\pi d^2$$

$$V=\frac{4}{3}\pi r^3 \qquad \text{or } V=\frac{1}{6}\pi d^3$$

These should be memorized so that time will not be lost in looking them up each time one is to be used.

ILLUSTRATIVE EXAMPLES

1. Find the number of square yards of silk necessary to cover a spherical balloon 50 feet in diameter.

Solution

Instruction	*Operation*
Step 1	**Step 1**
Apply the formula for finding surface area of a sphere	$S=\pi d^2$
Substitute the known values in the formula	$S=3.1416\times 50^2$ $=3.1416\times 2500=7854$
Surface area is 7854 sq. ft.	
Step 2	**Step 2**
Find the number of square yards in 7854 sq. ft.	$7854\div 9=872.7-$
872.7 sq. yd. Ans.	

2. The bottom of a cylindrical water tank is in the shape of a half sphere. The length of the cylindrical part is 22 feet and the diameter is 20 feet. How many gallons will the tank hold?

Solution

Instruction	*Operation*
Step 1	**Step 1**
Recall and use the rule for finding the volume of a cylinder. We must first find the area of an end of the cylinder	$A=\pi r^2$
Substitute the known values	$A=3.1416\times 10^2$ $=314.16$
Step 2	**Step 2**
Find the volume of the cylindrical part	Volume $=314.16\times 22$ $=6911.52$
Step 3	**Step 3**
Find the volume of the spherical part. It is a half sphere	$V=\frac{1}{2}$ of $\frac{1}{6}\pi d^3$
Substitute the given value for d and π	$V=\frac{1}{2}\times\frac{1}{6}\times 3.1416\times 8000$ $=2094.4$
Step 4	**Step 4**
Find the total volume of the tank Volume = 9005.92 cu. ft.	$6911.52+2094.4=9005.92$
Step 5	**Step 5**
Find the number of gallons (1 cu. ft. = $7\frac{1}{2}$ gals.) 67,544.4 gals. Ans.	$9005.92\times 7.5=67544.400$

PRACTICE PROBLEMS

1. How many square feet of canvas will cover a ball 5 feet in diameter? Ans. 78.54 sq. ft.

2. The outer diameter of a spherical shell is 12 inches. The inner diameter is 8 inches. Find the contents of the shell. Ans. 636.69 cu. in.

3. Find the weight of a cannon ball 15 inches in diameter if a cubic foot weighs 450 pounds. Ans. 460.19+lb.

4. How much material is wasted in cutting the largest possible sphere from a cube 10 inches on each side? Ans. 476.4 cu. in.

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on page 39.

If you miss more than two of the problems it means you should review the whole Section very carefully.

Do not try this trial examination until you have worked every practice problem in the Section.

Do not start the final examination until you have completed this trial examination.

Draw a sketch for each problem to help you visualize it.

1. Find the volume of a sphere whose radius is 25 inches.

2. The base of a certain prism is a right triangle whose two sides are $1\frac{1}{2}$ and 2 inches. The prism is 10 inches long. If the prism is placed in a cylindrical box 3 inches in diameter and 13 inches long, how many cubic inches can be used for packing material? (Use $\pi = \frac{22}{7}$.)

3. How many gallons of water will a circular vessel hold if the diameter of the bottom is 14 inches and of the top 21 inches, the depth of the vessel being 6 inches? (Use $\pi = \frac{22}{7}$.)

4. Assume a cylindrical pillar $3\frac{1}{2}$ feet in diameter and 20 feet high. The pillar stands with the 20-foot dimension vertical. Standing on top of the pillar is a cone whose base is 7 feet in diameter and whose slant height is 8 feet. The base of the cone rests on the top of the pillar. Find the cost of painting all exposed parts if painting costs 15 cents per square yard.

5. A cellar is 30 feet long and 24 feet wide. It is 8 feet deep. How many loads of one cubic yard each will have to be removed when this cellar is excavated?

6. A trough for watering cattle is 12 feet long, 3 feet wide, and 20 inches deep. Find the number of gallons it holds.

7. Find the lateral area of a cone whose base has a diameter of 18 inches and whose altitude is 24 inches.

8. A railroad water tank has the shape of a cylinder which stands on one end. The diameter of the cylinder is 30 feet and the height (or length) is 50 feet. If the tank contained water to a height of 20 feet, what was the volume in cubic feet of the empty portion of the tank?

9. The rainfall on a flat roof measuring 30 feet by 20 feet is 9.5 inches during a certain period. How much does this amount of rain weigh?

10. A gasoline container, in the shape of a cylinder, had a radius of 12 inches and a height of 48 inches. It was full of gasoline. A man purchased one third of the gasoline. How much must he pay if gasoline costs $0.20 a gallon?

FINAL EXAMINATION

Draw a sketch for each problem. Draw the sketches neatly and make them fairly accurate.

1. What is the volume of a square pyramid whose base is 7 feet on each side and whose height equals the diagonal of the base?

2. Find the surface of a cube whose edge is 6 inches.

3. Assume a box which is 4 feet square. If a large ball having a diameter of 2 feet is placed inside the box, how much space, in cubic feet, is left for packing material?

4. What is the total area of a prism whose base is a square 15 feet on each side, the altitude of the prism being 35 feet?

5. A cylindrical tank holds 450 gallons of water. What is the diameter of the tank if its length is 6 feet?

6. Find the volume of a hexagonal pyramid whose base perimeter is 54 inches and whose height is 8 feet.

7. A cone is 10 feet high and has a base diameter of 6 feet. A frustom of a cone has an altitude of 8 feet. The radius of its top is 8 feet and its bottom 12 feet. Find the difference in volume, in cubic inches, between the cone and the frustom.

8. A cone measures 6 inches in diameter at the base. The distance from the edge of the circumference to the top is 8 inches. Find its volume.

9. *A* and *B* bought a solid ball of twine 8 inches in diameter for $1.00. *A* unwound from the outside until the diameter of the part that was left was 4 inches. How much should each pay?

10. A cylinder is 12 inches in diameter and 16 feet long. The material it is made of weighs 10 pounds per cubic foot. If a hole having a diameter of 5 inches is bored through the cylinder from end to end, what is the weight of the remaining material?

Hint: First find the weight of the cylinder before the hole was bored. Next find weight of material removed by the hole.

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. This solution requires the use of **Rule 14.** But we cannot use **Rule 14** until we have found the surface by **Rule 13.**

Following **Rule 13**, if $25 = \text{radius}$,

$$\text{Diameter} = 2 \times 25 = 50$$
$$\text{Diameter squared} = 50 \times 50 = 2500$$

Then

$$3.1416 \times 2500 = 7854.0000 = \text{surface}$$

Now following **Rule 14,**

$$\tfrac{1}{3} \text{ of radius} = \tfrac{1}{3} \text{ of } 25 = 8.33+$$

Then

$$7854 \times 8.33 = 65423.82$$

Volume = 65,423.82 cubic inches. Ans.

2. We must find the volume of the prism and also of the cylindrical box and then simply subtract one from the other.

The base of the prism is a right triangle having sides $1\frac{1}{2}$ and 2 inches. We must use **Rule 3** to find the volume of this prism. To do that we need the area of the base. In Section 12 we learned that the area of a triangle is the product of $\frac{1}{2}$ the base times the altitude. We can assume that the 2-inch side of the triangle is the base and the $1\frac{1}{2}$-inch side is the altitude. Then

$$\frac{1\frac{1}{2} \times 2}{2} = 1.5 \text{ square inches, area of base of prism}$$

Following **Rule 3**, we multiply the area of the base of the prism by its altitude. Its altitude is 10 because the prism is 10 inches long. Then

$$1.5 \times 10 = 15 \text{ cubic inches, volume of prism}$$

Next we must find the volume of the cylindrical box. **Rule 5** says the volume of a cylinder is found by multiplying the area of its base by its altitude.

If the cylinder is 3 inches in diameter, we use the principles given in Section 12 to find the area of its base. We assume the cylindrical box is standing on one of its ends. Thus

$$\text{Area of base} = \pi\left(\tfrac{3}{2}\right)^2 = \tfrac{22}{7} \times \tfrac{9}{4} = 7.07 \text{ square inches}$$

The cylinder is 13 inches long, so by **Rule 5**

$$7.07 \times 13 = 91.9+ \text{ cubic inches, volume of cylinder}$$

Then

$$91.9 - 15 = 76.9 \text{ cubic inches. Answer.}$$

3. If you will carefully draw a sketch of the vessel, you will find it to be like Fig. 21 in your text, except that your sketch will look like Fig. 21 turned upside down.

We must first find the volume of the vessel in cubic inches, then we can divide by 231 (see Section 7) to find the number of gallons.

We use **Rule 12.**

Both bases are circles; one is 14 inches and the other 21 inches in diameter. We learned in Section 12 that the area of a circle equals πr^2. The radius of the 14-inch diameter base is 7 inches, and the radius of the 21-inch diameter base is 10.5 inches. Thus

$$\text{1st base area} = \tfrac{22}{7} \times 7^2 = 154 \text{ square inches}$$
$$\text{2nd base area} = \tfrac{22}{7} \times 10.5^2 = 346.5 \text{ square inches}$$

To use **Rule 12** we must find the sum of the base areas, thus

$$154 + 346.5 = 500.5$$

To this sum we must add the square root of the product of the two bases,

$$\sqrt{154 \times 346.5} = 231 \text{ square inches}$$

(See Section 8 for explanation of roots.)

Then $$500.5 + 231 = 731.5 \text{ square inches}$$

The altitude is 6 inches. Then $\frac{1}{3}$ of $6 = 2$ and

$$731.5 \times 2 = 1463 \text{ cubic inches, volume of vessel}$$

There are 231 cubic inches in a gallon. Then

$$1463 \div 231 = 6\tfrac{1}{3} \text{ gallons} \quad \text{Ans.}$$

4. In this problem there are three areas to be painted: (*a*) The lateral area of the cylinder, (*b*) the lateral area of the cone, and (*c*) that part of the base of the cone that is exposed as it stands on the top of the pillar which is only $3\frac{1}{2}$ feet in diameter.

(*a*) *Lateral Area of the Cylinder*

First find the circumference.

Circumference $= \pi \times$ diameter (Section 12, page 32)

$$C = \tfrac{22}{7} \times 3.5 = \tfrac{77}{7} = 11 \text{ feet}$$

Lateral area of a cylinder equals its circumference multiplied by the altitude. Then

$$11 \times 20 = 220 \text{ square feet, area of cylinder}$$

Neither the cylinder top nor base will be painted because they are not exposed.

(*b*) *Lateral Area of Cone*

Lateral area of a cone is found by multiplying the circumference of base by half the slant height. First find the circumference of the base of the cone.

$$C = \pi \times \text{diam.}$$
$$C = \tfrac{22}{7} \times 7 = \tfrac{154}{7} = 22 \text{ feet}$$

Then $$22 \times 4 = 88 \text{ square feet, lateral area of cone}$$

(*c*) *Exposed Area of Base of Cone*

Look at the sketch you have drawn to illustrate this problem and you will see that the area of the base of the cone, minus the area of the top of the cylinder, will be the exposed area of the base of the cone.

We know the circumference of the base of the cone (having previously found it when we calculated the lateral area) so we will use Formula (14) of Section 12, $A = C \times \frac{r}{2}$.

The radius is $\frac{1}{2}$ of $7 = 3\frac{1}{2}$ or 3.5. Then, substituting, we have

$$A = 22 \times \frac{3.5}{2} = 22 \times 1.75 = 38.5$$

Then 38.5 square feet is the entire area of base of cone.

We use the same formula to find the area of the top of cylinder because we have already found its circumference. The radius is $\frac{1}{2}$ of $3\frac{1}{2} = 1.75$

Substituting,

$$A = 11 \times \frac{1.75}{2} = 11 \times .875 = 9.625$$

Then 9.625 square feet is the area of the top of the cylinder, which is to be subtracted from area of base of cone.

$$38.5 - 9.625 = 28.875 \text{ square feet}$$

Then 28.875, call it 28.88 square feet, is the exposed area of base of cone.

Having calculated the three exposed areas, we must add them. Thus

$$\begin{array}{r} 220 \\ 88 \\ 28.88 \\ \hline 336.88 \end{array} \text{ square feet}$$

We divide this by 9 to find the number of square yards.

$$336.88 \div 9 = 37.43 \text{ square yards}$$

$$37.43 \times .15 \text{ per square yard} = \$5.6145 \text{ or } \$5.61+ \quad \text{Ans.}$$

5. The shape of this cellar is rectangular. In other words the shape is that of a prism. We want to find the volume in cubic yards.

Rule (3) on page 9 says that the volume of a prism is found by multiplying the base area by the altitude. In Section 12 you learned that the area of a rectangle is found by multiplying the two given dimensions. In this problem the base of the prism is 30 feet by 24 feet. Then $30 \times 24 = 720$ square feet, which is the base area. Having the base area we can multiply it by the altitude (8). This is really the depth of the cellar. Then

$$720 \times 8 = 5760 \text{ cubic feet}$$

You will note that when the area of the base of a rectangle or prism (square feet) is multiplied by the altitude, the product is volume in cubic feet.

We want to know how many cubic yards will have to be removed to complete the digging or excavation. There are 27 cubic feet in one cubic yard. Thus

$$5760 \div 27 = 213\frac{1}{3} \text{ loads.} \quad \text{Ans.}$$

The above solution shows that volume of a solid the shape of Fig. 8B, page 9, can be found by multiplying the base area by the altitude or height.

6. In Section 7 you learned that $7\frac{1}{2}$ gallons equal one cubic foot. Or, to fill a space equal to one cubic foot requires $7\frac{1}{2}$ gallons of a liquid such as water.

The trough is in the form of a rectangle or prism. The base dimension is 12 feet by 3 feet. Thus the base area is $12 \times 3 = 36$ square feet. The depth (altitude) is 20 inches $= 1\frac{2}{3}$ feet. Then

$$36 \times 1\tfrac{2}{3} = 60 \text{ cubic feet}$$

Next, multiplying 60 by $7\frac{1}{2}$ gives 450 gallons. Answer.

7. To find the lateral area of a cone, according to the rule on page 26, we have to know the circumference of the base and the slant height.

We can find the circumference by multiplying 3.1416 by 18.

$$3.1416 \times 18 = 56.5488 \text{ inches; call it } 57. \text{ (Section 5)}$$

To find the slant height we make use of the right triangle law you studied in Section 12. Refer to Fig. 19, page 25 in this section. Assume this figure applies to our problem. We know that the altitude (line AB) is 24 inches. The diameter is 18 inches, so BC is equal to $\frac{1}{2}$ of 18 or 9 inches. We want to find AC, the hypotenuse (slant height).

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 24^2 + 9^2 \\ &= 576 + 81 \\ AC^2 &= 657 \\ AC &= \sqrt{657} \\ AC &= 25.63 \end{aligned}$$

We need to know $\frac{1}{2}$ of slant height. So $\frac{1}{2}$ of $25.63 = 12.81$.

Circumference is 57 and $\frac{1}{2}$ slant height is 12.81, then

$$57 \times 12.81 = 730.17 \text{ or } 730+ \text{ sq. inches. Ans.}$$

8. **Rule 5** says the volume of a cylinder is found by multiplying the area of its base by its altitude.

If the tank has 20 feet of water in it, then the altitude of the empty portion of the tank is $50 - 20 = 30$ feet. Therefore if we find the volume of a cylinder 30 feet in diameter and having an altitude of 30 feet, we will have the required answer.

One way to find the area of the base (a circle) is by multiplying 3.1416 by radius squared. The radius is half the diameter, thus the radius equals $\frac{1}{2}$ of $30 = 15$ feet. Then $15^2 = 225$, and $3.1416 \times 225 = 706.8600$ or 706.86 square feet.

Volume equals $706.86 \times 30 = 21205.80$ cubic feet. Answer.

9. The roof is rectangular so if we multiply its area by the depth of the rainfall we will have the volume of the water.

$30 \times 20 = 600$ square feet (area). We will change this to square inches because the depth (altitude) is given in decimal inches.

$$600 \times 144 = 86{,}400 \text{ square inches (area)}$$
$$86400 \times 9.5 = 820{,}800 \text{ cubic inches.}$$

One cubic foot of water weighs $62\frac{1}{2}$ pounds (given in Section 7), thus we

must change 820,800 cubic inches to cubic feet. There are 1728 cubic inches in 1 cubic foot.

$$820{,}800 \div 1728 = 475 \text{ cubic feet}$$

Then $475 \times 62\frac{1}{2} = 29{,}687.5$ pounds. Answer.

10. Before we can get very far in the solution of this problem we must find the number of gallons the container holds. We know that 231 cubic inches equals 1 gallon. Thus the first step is to find the volume of the container in cubic inches. Use **Rule 5**, page 15.

Base area is $A = \pi \times r^2$

$\pi = 3.1416$

$r^2 = 12 \times 12 = 144$

Substituting $A = 3.1416 \times 144 = 452.3904$ square inches

Following the rule, we multiply this base area by the height.

$$452.3904 \times 48 = 21714.7392 \text{ cubic inches}$$

which is the volume. By the rules you learned in Section 5 we can shorten this to 21,714.74.

Next change 21,714.74 cubic inches to gallons.

$$21714.74 \div 231 = 94+ \text{ gallons}$$

The man purchased one third of this amount at $.20 per gallon, so

$\frac{1}{3}$ of $94 = 31.33$ gallons

$31.33 \times .20 = \$6.27$ Answer.

SECTION 14

LESSON OUTLINE FOR LOGARITHMS

Every good teacher has a definite plan for carrying out his work, and he leads the student step by step through this plan, although the student may not be conscious of the plan. However, since you and your instructor will not be meeting each other personally, we want to tell you something about our plan of teaching Logarithms. We have divided this Logarithm Section into a series of lessons. Each lesson has a definite aim, and it is finished when the student accomplishes the aim. Every lesson has four steps:

Step 1—The preparation or background
Step 2—The presentation of the new material
Step 3—The application or trial
Step 4—The test

Lesson 1

This lesson commences with the beginning of the Section, page 1, and ends at "Finding Number that Corresponds to a Logarithm," page 9. For Step 1, recall what you know about multiplication and division in Arithmetic and about roots, powers, and exponents. For Step 2, learn the meaning of logarithms and its applications and how to find the logarithm of a number. For Step 3, find the logarithm of 220 as illustrated in this book. For Step 4, find the logarithm of 221, of 222, of 222.9.

Lesson 2

This lesson commences with "Finding Number that Corresponds to a Logarithm," page 9, and ends at "Multiplication," page 12. For Step 1, recall what you have learned in Lesson 1. For Step 2, learn the method of finding a number that corresponds to a given logarithm. For Step 3, find the number corresponding to 3.203848 as given in the book. For Step 4, find the number corresponding to −4.331690 as given in the book.

Lesson 3

This lesson commences with "Multiplication," page 12, and ends at "Division," page 22. For Step 1, recall what you have just learned in the two previous lessons. For Step 2, learn the method of multiplying by logarithms. For Step 3, work Illustrative Examples 1, 2, 3, and 4. For Step 4, work the Practice Problems.

Lesson 4

This lesson commences at "Division," page 22, and ends at "Multiplication and Division," page 25. For Step 1, recall what you have just learned in the three previous lessons. For Step 2, learn the method of dividing by logarithms. For Step 3, work Illustrative Example, 5. For Step 4, work the Practice Problems.

Lesson 5

This lesson commences with "Multiplication and Division," page 25, and ends at "Roots and Powers," page 26. For Step 1, recall what you have learned in

the four previous lessons. For Step 2, learn the method of multiplying and dividing by logarithms. For Step 3, work Illustrative Example 6. For Step 4, work the Practice Problems.

Lesson 6

This lesson commences with "Roots and Powers," page 26, and ends at "Decimal Numbers with Exponents," page 30. For Step 1, recall what you have just learned in the five previous lessons. For Step 2, learn the method of finding roots and powers by logarithms. For Step 3, work Illustrative Examples 7 and 8. For Step 4, work Practice Problem 1.

Lesson 7

This lesson commences with "Decimal Numbers with Exponents," page 30, and ends at "Combination of Operations," page 32. For Step 1, recall what you have just learned in the six previous lessons. For Step 2, learn the method of raising decimal numbers to powers by logarithms. For Step 3, find the logarithm of $.07854^3$, as shown on page 30. For Step 4, work Illustrative Example 9.

Lesson 8

This lesson commences with "Combination of Operations," page 32, and ends at the examination. For Step 1, recall what you have learned in the seven previous lessons. For Step 2, learn the method of combining all the operations of the previous lessons. For Step 3, work Illustrative Example 10. For Step 4, work the following example: $\dfrac{302 \times 193}{824 \div 26}$

LOGARITHMS

This is a day of efficiency and time-saving devices. Anyone who can devise some means for saving time on a job is well paid for his services. Competition is so great in business that time-saving devices are looked for continually. And as it is in business so it is in our study. We need every help we can get which will make our time count to the greatest advantage.

A shortcut method, called logarithms (pronounced log′a-rĭthms), has been devised which will not only save time in multiplying large numbers together, but it can also be used to divide, to extract the square root, the cube root, or any other root; to square a number, cube it, or figure out the exact amount for any other power of the given number; or it can be used for any combination of these operations. Take, for example, the following problem.

Find the value of $\sqrt{\dfrac{3678^4 \times .03257^2}{1679^3 \times 1.345^5}}$

The symbol in front of the problem is called the **square root** or **radical sign.** It means that the square root of the combined operations under the symbol is to be extracted. To find the square root of a given number means to find a number which when multiplied by itself will equal the given number. For instance, the square root of 9 is 3, for 3 times 3 equals 9.

The first number in the numerator is 3678. It has a small figure above it. This figure is called an **exponent** and means that the number is taken as many times as the exponent indicates. Each number in this problem has an exponent, and therefore each number is taken as a factor as many times as its exponent indicates.

To solve this problem we would have to multiply the first number in the numerator (3678) four times (that is, $3678 \times 3678 \times 3678 \times 3678$); then we would have to multiply the second number (.03257) two times (that is, $.03257 \times .03257$); and then we would have to multiply the results of these two operations together, which

would give us the combined numerator. Similarly, for the denominator we would have to multiply the first number in the denominator (1679) three times (that is, $1679 \times 1679 \times 1679$); we would have to multiply the second number in the denominator (1.345) five times; and then we would have to multiply these two results together to get the combined denominator. After we have done this, we would have to divide the combined numerator by the combined denominator. We would now have one number, which is the combination of all the terms under the **square root sign.** The final operation would be to extract the square root of this number.

You, no doubt, already appreciate how long a problem this would be to work out by arithmetic. When you have learned how to use the logarithmic tables, we will show you in detail just how to work a similar problem by logarithms and you will see how short it is. This short method also takes away the chance of many errors which are constantly being made in the operations of multiplying and dividing, therefore we feel sure that you will want to learn this method right away.

The invention of logarithms has been accorded to John Napier, who was a baron of Scotland. The tables, however, which we are using at the present time are called the Briggs Tables. Henry Briggs was a professor of geometry in London, England. He often visited Mr. Napier in order to get his ideas on this subject of logarithms. Then Briggs spent a large part of his time from 1615 to 1628 in compiling his tables.

The fact that it took one man a great many years to calculate and produce these tables is sufficient proof that it was a real job, and because a man was willing to spend years to produce a tool with which to make your work in Mathematics easier and shorter, you should be glad to use it. The important thing at this time, however, is for you to **learn how** to use this tool.

Read the following table carefully and note the exponents:

$10^1 = 10$ and the logarithm of 10 is 1
$10^2 = 100$ and the logarithm of 100 is 2
$10^3 = 1000$ and the logarithm of 1000 is 3
$10^6 = 1000000$ and the logarithm of 1000000 is 6
$10^x = N$ and the logarithm of N is x

In the last line x **is the index of the power to which 10 must be raised in order to equal** N, or x is **the logarithm of N to the base 10.**

Therefore, *the common logarithm of a number is the exponent to which* 10 *must be raised to give the number.*

The Logarithmic Tables give a systematic list of exponents of 10 with their corresponding numbers.

If a decimal exponent is used, then the logarithm has a decimal part, which is written in decimal form.

Thus $10^{1.6}=39.81+$ and the logarithm is 1.6.

Thus, it is seen that when the number is an exact power of 10, the logarithm has no decimal part; and when the number is not an exact power of 10, the logarithm has a decimal part. These decimal parts are the parts which required such a long time for Mr. Briggs to calculate. You can multiply a number two or three times quite easily, but you cannot multiply it $2\frac{1}{2}$ or $2\frac{1}{3}$ times.

There are two parts to every logarithm. The first part, which is to the left of the decimal point, as figure 2 in 2.342817, is called the **characteristic.** The second part, which is to the right of the decimal point, as figures .342817, is called the **mantissa.** These are large words but you will soon learn to handle them easily. Characteristic is pronounced char′ac-ter′is-tic. Mantissa is pronounced man-tis′sa.

CHARACTERISTIC OF A LOGARITHM

A characteristic may be either positive or negative. Its value depends entirely on the location of the decimal point in the original number. To get the characteristic for numbers which have figures to the left of the decimal point the following rule will apply:

Count the number of figures to the left of the decimal point and subtract 1 from this total and you have the characteristic of the number.

Some numbers are given in tabulated form in Table I to show you how this rule operates.

In the same way as shown in Table I, there can be any number of figures to the **left** of the decimal point and by **subtracting one** from the number of figures, you get the characteristic. These

TABLE I

Number	Figures to left of Decimal Point	Minus	1	Equals	Positive Characteristic
2.34567	1	−	1	=	0
23.4567	2	−	1	=	1
234.567	3	−	1	=	2
2345.67	4	−	1	=	3
23456.7	5	−	1	=	4

characteristics are all **positive.** Cover answers with a sheet of paper, then pick out the characteristics for the following numbers:

Number	Answer	Number	Answer
1.23456	0	34,567,890.	7
654,321.0	5	230,000.00	5
897.654	2	8,760,007.2	6
37.09843	1	92,007,000.	7
2768.91	3	1.0007239	0
7.0	0	900.	2
100.	2	100,000,000.	8

When the figures are all to the **right** of the decimal point, the characteristic is **negative.** When the first figure to the **right** of the decimal point is **not a cipher,** the characteristic is a **minus one**

TABLE II

Number	Ciphers to right of Decimal Point	Plus	1	Equals	Negative Characteristic
.234567	0	+	1	=	-1 or $\bar{1}$
.023456	1	+	1	=	-2 or $\bar{2}$
.002345	2	+	1	=	-3 or $\bar{3}$
.0000234	4	+	1	=	-5 or $\bar{5}$
.0000023	5	+	1	=	-6 or $\bar{6}$

(-1), regardless of the number of other figures to the right. When only the first figure to the right of the decimal point is a **cipher,** the

characteristic is a **minus 2** (−2), and for each **additional cipher** to the right of the decimal point add **one** to the next preceding characteristic and place a minus sign in front or above the result. This is illustrated in Table II.

In the same way as shown in Table II, there can be any number of ciphers to the **right of the decimal point,** and by adding one to the number of ciphers and placing a − sign in front or above this sum, you get the characteristic. These characteristics are all **negative.** Cover answers with a sheet of paper, then pick out the characteristics for the following numbers:

Number	Answer	Number	Answer
0.3	$\bar{1}$	0.010203	$\bar{2}$
.0005	$\bar{4}$	.001002	$\bar{3}$
00.6789	$\bar{1}$	.0000009	$\bar{7}$
.00803	$\bar{3}$	00.0002003	$\bar{4}$
.00001	$\bar{5}$	000.0000823	$\bar{5}$

The complete rule for obtaining either a positive or a negative characteristic may be stated as follows:

When the number is one or greater than one (1), *the characteristic is positive and is one less than the number of figures to the left of the decimal point. When the number is less than one, the characteristic is negative and is one more than the number of ciphers between the decimal point and the first significant figure.*

The characteristic of a number cannot be given in the tables because it depends entirely on the location of the decimal point.

MANTISSA OF A LOGARITHM

The mantissa is entirely independent of the decimal point. This is just the opposite of the characteristic. The mantissa is always positive and is always a decimal number. In other words it is the decimal part of a logarithm.

If you have two or more quantities composed of the same significant figures in the same order, it does not make any difference with the mantissa how many ciphers are to the right or to the left of the significant figures. The mantissa is the same in every case.

Let us study the Logarithmic Tables and find out just what is meant by the above statement. The following illustration is the first horizontal line on page 2 of the Logarithmic Tables.

LOGARITHMIC TABLES

N	0	1	2	3	4	5	6	7	8	9	D
100	000000	000434	000868	001301	001734	002166	002528	003029	003461	003891	432

The first vertical column in the Logarithmic Tables is headed by the letter N, which stands for the word **number.** This column contains the numbers whose mantissae are given in the horizontal lines corresponding with the numbers. Thus in the above illustration 100 is shown under N and the various mantissae for 100 with the following additional tenths (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) are shown in the horizontal line with 100 and in the vertical columns to correspond to the tenths given. To explain further, the mantissa for 100.0 is .000000 and the characteristic is 2, as shown on page 2 of the text. The mantissa for 100.1 is .000434, as found in the column headed by **1**. The mantissa for 100.2 is .000868, as found in the column headed by **2**. Thus as the tenths are added to 100, the mantissa increases until under the column headed by **9**, the mantissa for 100.9 is given as .003891.

The equally spaced steps, one tenth apart, are for aiding the reader to obtain accurate results quickly. In the Logarithmic Tables the second number in column N is 101. If the intervening 10 steps between 100 and 101 were not given, it would take considerable time to calculate them.

As the numbers in column N progress to the bottom of the page, one unit is added as each line is passed. Thus from the first mantissa in column **0** to the last mantissa of column **9**, a complete set of mantissae are given for each 1/10 interval for the numbers given in column N.

The Logarithmic Tables do not show any decimal points. As the mantissae are entirely decimal, the decimal point will be at the left of each left-hand figure.

In column N the decimal point can be placed at any place to suit the user. For instance, the first number may be 0.10, 1.00, 10.0,

100, 1000, 10000, or any other multiple of ten. The mantissa is the same in any case as the only significant figure is 1. The only difference is in the characteristic, as previously shown.

In the last column to the right in the Logarithmic Tables, headed **D**, the differences between any two adjoining mantissae in each horizontal line are given. Since there are ten mantissae in each line, the differences are not always the same; therefore, column **D** gives the average difference for each line. This variation does not make much difference ordinarily. This column is given to save you the labor of subtracting and also insures accurate work. The use for these differences as given in column **D** will be explained later.

HOW TO FIND THE LOGARITHM OF A NUMBER

From the accompanying illustration, copied from page 4 of the Logarithmic Tables, we will show you how to find the mantissa for any given number. Take number 220 in column **N.** Let us follow this line right across horizontally and see what the different mantissae are for 220, for 220.1, for 220.2, for 220.9, for 221, etc. See Table III.

LOGARITHMIC TABLES

N	0	1	2	9	D
220	.342423	.342620	.342817	.344196	197
221	.344392	.344589	.344785	.346157	196
222	.346353	.346549	.346744	.348110	195

TABLE III

Number	Mantissa	Column found in	Characteristic
220.0	.342423	0	2
220.1	.342620	1	2
220.2	.342817	2	2
220.9	.344196	9	2
221.0	.344392	0	2
222.2	.346744	2	2

In the vertical column under **0**, first line, we find the mantissa .342423. This is in line with number 220 in column **N** and as there

are no tenths added in the **0** column, this is the proper mantissa for number 220.0. There are three figures to the left of the decimal point; by subtracting 1, according to Table I, we get the characteristic 2. Therefore, the complete logarithm for 220.0 is 2.342423.

In the next vertical column under **1**, first line, we find the mantissa .342620. This mantissa is larger than the one in column **0** in the same line because we have added one-tenth (0.1) to our number 220.0, making it 220.1. The characteristic has not changed, so our complete logarithm for 220.1 is 2.342620.

In the next vertical column under **2**, first line, we find the mantissa .342817. This has increased over the preceding mantissa because we have added one-tenth to our number 220.1, making it 220.2. The characteristic, however, remains 2 for we still have the same number of integers to the left of the decimal point in the number, so the logarithm for 220.2 is 2.342817.

As we progress to the right across the table we add 0.1 to the number shown in the next preceding column, and when we reach the column under **9**, the number becomes 220.9 and the mantissa given is .344196.

If we add 0.1 to this number 220.9, we have 221.0, which is the number given in the second line under **N**, and the mantissa is the second in the column under **0**.

Thus you see the table progresses in 0.1 steps from the number 220 until the bottom of the page is reached in the column under **9**. On this page 4 of the Logarithmic Tables, the last number under **N** is 250, so the mantissa for 250.9 is the last one in the column under **9**. Thus the complete logarithm for 250.9 is 2.399501. Therefore, to find the mantissa for any number from 200 to 250.9 you would use page 4 of the Logarithmic Tables.

These same numbers (220, 220.9, 221, etc.) may have the decimal point changed either to the right or to the left without changing the mantissae at all, but the characteristics will be changed. This is illustrated in Table IV.

In Table IV we have used the numbers as given in Table III, but have changed the decimal points in order to show you how the characteristics change but the mantissae are the same and do not change. Apply the rule which we gave you concerning the characteristic and refer to Tables I and II, and you will understand

TABLE IV

Number	Logarithm	Characteristic				
2.200	0.342423	1	−	1	=	0
22.01	1.342620	2	−	1	=	1
220.2	2.342817	3	−	1	=	2
2209.	3.344196	4	−	1	=	3
0.221	$\bar{1}$.344392	0	+	(−1)	=	−1 or $\bar{1}$
0.0221	$\bar{2}$.344392	−1	+	(−1)	=	−2 or $\bar{2}$
.00221	$\bar{3}$.344392	−2	+	(−1)	=	−3 or $\bar{3}$

Table IV. To illustrate, note that the last three numbers in Table IV have the same consecutive figures (221), but the decimal point has been moved one figure more to the left in each case. Therefore, while these numbers have the same mantissa, the characteristic has become negative and changed according to the number of ciphers to the right of the decimal point.

FINDING NUMBER THAT CORRESPONDS TO A LOGARITHM

In using the Logarithmic Tables it is also necessary to know how to find the number when the logarithm is given. Take, for example, logarithm 3.203848. To find the number which corresponds to this logarithm, we just reverse the process for finding the logarithm for the number.

Take the first three figures of the mantissa, which are 203, and turn to the Logarithmic Tables. Follow down the first vertical column **0** until we find nearly the same mantissa as the logarithm contains. Then follow across the horizontal line when necessary to get the same, or nearest to but less, mantissa than the one in the problem. We will find this mantissa in the tenth line of column **9**, page 3. In this particular case, we find the exact mantissa which we have in our logarithm (Step 1).

The number which corresponds to this mantissa is given in the same line by the three figures in column **N**, which are 159, and we add on to the right end figure 9 from the column under **9**. This gives the number 1599.

It is better to use a smaller mantissa than a larger one. This is shown later.

Step 1

LOGARITHMIC TABLES

Page	Line	N	9	D
3	Tenth	159	.203848	272

Annex, or place, after 159 the figure 9 which makes the number 1599. 3+1=4, so point off four figures to left of the decimal point.

To find where to point off the figures for the decimal point, we **reverse** the operations used in finding the characteristic. In our problem we have a characteristic of 3, to which we add 1, which makes 4. Begin at the left and count off four figures and place the decimal point.

Having a positive characteristic, the rule is: *Add one to the given characteristic to find the number of figures to the left of the decimal point.*

Having a negative characteristic, the rule is: *Subtract one from the negative characteristic to find the number of ciphers to the right of the decimal point.*

To illustrate these rules a number of logarithms and the way the corresponding numbers are pointed off for both the positive and the negative characteristic are given in Table V.

TABLE V—Locating the Decimal Point

Logs. +Characteristic	To Characteristic add one	Figures to left of point	Number
0.301030	0 + 1 = 1	One	2.00
1.204120	1 + 1 = 2	Two	16.00
2.161368	2 + 1 = 3	Three	145.00
3.203848	3 + 1 = 4	Four	1599.00
5.687013	5 + 1 = 6	Six	486421.4 (Prob. I)
Logs. −Characteristic	From characteristic subtract one	Ciphers to right of point	Number
$\bar{1}.518514$	1 − 1 = 0	None	.330
$\bar{2}.521138$	2 − 1 = 1	One	.0332
$\bar{3}.653502$	3 − 1 = 2	Two	.004503
$\bar{4}.660865$	4 − 1 = 3	Three	.000458
$\bar{4}.331690$	4 − 1 = 3	Three	.00021462+

Let us now take a logarithm with a minus characteristic and follow it through in the same way as we did the logarithm with the positive characteristic. For example: We will take the last one in Table V. The logarithm has a minus 4 characteristic, and the mantissa is 331690. The minus sign is placed above the characteristic so that it will not be interpreted as a minus logarithm, for a minus sign in front of a number refers to all of the figures in the number following it. Take the first three figures of the mantissa, which are 331. Follow down through the column **0** on Page 4 of the Logarithmic Tables until we find the mantissa which contains the figures 330, which are the nearest to 331 in this column. In line 15, corresponding to number 214 in column **N,** we find these figures and by following this line over to vertical column **6,** we find the nearest mantissa to the one we have and a little less. You will note by looking under column **7** at the next mantissa that it is quite a little bit over. (Step 2.)

Step 2

LOGARITHMIC TABLES

Page	Line	N	6	7	D
4	15	214	.331630	.331832	202

As before, we have three figures, 214, under the vertical column **N,** and to the right of that we annex figure 6 from column **6,** and that gives the four significant figures 2146. To this we must annex the difference of the mantissa in the table and our logarithm for the mantissa is a little less than the mantissa in the problem.

Step 3 $\bar{4}.331690$ our logarithm

$.331630$ our logarithm in the table

Difference = 60

In Step 3 we subtract the mantissa in column **6** from our logarithm. It gives us a difference of 60. We now have the question of how much to annex to the number 2146 for this difference of 60 in the mantissae. You will note the difference between the mantissae in columns **6** and **7** (Step 2) is the value given in column **D,** or 202. In other words, if we would use column **7** instead of column **6,**

the mantissa would have to be 202 more. As we have only 60 more than in column **6**, we divide 60 by 202, which equals 29+, or the fractional part which we annex to our number. This is shown in Step 4.

Step 4 $\frac{60}{202}=29+$

Annexing 29+ to 2146 we get 214629+
$4-1=3$ (Table V). Placing 3 ciphers to right of decimal point $=.000214629+$ Answer

MULTIPLICATION

In order to multiply two numbers by the use of logarithms, we find the logarithms of the two numbers and then add these two logarithms together. This sum is the logarithm of the product of the two numbers. Then we turn to the Logarithmic Tables and find the number which corresponds to this logarithm. We will take two of the numbers and their logarithms from Table III to illustrate.

NOTE: In problems use log, abbreviation for logarithm, and mant., abbreviation for mantissa.

Problem 1. Multiply 220.2 by 2209

Instruction	*Operation*	
Step 1	**Step 1**	
Find log of the first number in problem.	Find log of 220.2	
Do this by finding the mantissa of the number.	Mantissa of 220.2	= .342817
Next find the characteristic of the number.	Characteristic of 220.2	=2.
Combine and you have the log of the number.	Log of 220.2	=2.342817
Step 2	**Step 2**	
Find log of the second number in problem.	Find log of 2209	
Do this by finding the mantissa of the number.	Mantissa of 2209	= .344196
Next find the characteristic of the number.	Characteristic of 2209	=3.
Combine and you have the log of the number.	Log of 2209	=3.344196

Step 3

To multiply numbers, add the logs of those numbers together.

Step 3

Add log of 220.2 and log of 2209 together.

From Step 1, log of 220.2 = 2.342817
From Step 2, log of 2209 = 3.344196
Log of product = 5.687013

Step 4

Find number that corresponds to this logarithm.

Do this by finding the number in Logarithmic Tables that has a mantissa the same as the mantissa in your problem, or slightly LESS. Subtract this mantissa from the given mantissa.

Step 4

Find number that corresponds to given mantissa.

From Step 3, mantissa = .687013
Mantissa of 4864 = .686994
Difference = 19

486 from column N
4 from column 4
4864 = first four figures of number

LOGARITHMIC TABLES

Page	Line	N	4	5	D
9	37	486	.686994	.687083	89

Step 5

Find difference between mantissa in adjacent columns on same horizontal line in the log tables or use column D.

Step 5

Mantissa of 4865 = .687083
Mantissa of 4864 = .686994
Difference = 89
Column D = 89

tep 6

Divide difference in Step 4 by difference in Step 5.

Step 6

Divide 19 by 89

```
89 ( 19.0 ) .214
     17 8
      1 20
        89
       310
```

Step 7

Place the quotient of Step 6 at the right-hand end of the number found from the table, Step 4.

Step 7

Number in Step 4 = 4864
Quotient in Step 6 = 214–
Result = 4864214

Step 8

Locate decimal point.
Number of figures to left of decimal point is characteristic +1.
Begin at left of result in Step 7 and count six figures and place point.

Step 8

Characteristic of log is 5+1=6 (Table V)
Decimal point is between figure 1 and figure 4
Thus 486421.4
Therefore 220.2 multiplied by 2209 is 486421.4

The problem will now be given in condensed form to show the actual work.

Multiply 220.2 by 2209.

Step 1	Log of 220.2	=	2.342817
Step 2	Log of 2209	=	3.344196
Step 3	Log of product	=	5.687013
Step 4	Mant. of 4864	=	.686994
	Difference	=	19
Step 5	Difference	=	89
Step 6	19÷89	=	.214
Step 7	214 annexed to 4864	=	4864214
Step 8			486421.4 Answer

In order that you may understand all the details, we will describe this first problem carefully.

We have taken the two logs from Table II, set them down under each other, and added them and thus obtained the logarithm of a product. Now to find the number which corresponds to this logarithm of the product, we reverse the process for finding the logarithm. We turn to our Logarithmic Tables, and look for the mantissa which corresponds to the mantissa of the product.

To do this, take the first three figures of the mantissa of the product (in this case 687), and look for the same figures in a column in the Logarithmic Tables. You will find these figures on page 9 in the horizontal line where column N equals 486. By following across horizontally until you get to the vertical column 4, you will find the same mantissa which we have shown you in the solution, Step 4.

The figures 486 in column N are the first three figures of your new number, and the figure 4, at the head of column 4, is the fourth figure of your number.

These four figures we have placed in Step 4, in front of the mantissa which we have just found. This is the **nearest lower mantissa** to the one which we have in our logarithm for the product of the two numbers. We now subtract this mantissa from the one above it, as shown in Step 4. The difference is 19. This means that our number is a little larger than this mantissa, so now we must do what we call interpolate (pronounced ĭn-tûr′pō-lāte). If you look in column **D**, Step 4 of our problem, you will find there is a difference between the mantissa in column **4** and the one in column **5** of 89. So we must divide the difference of the mantissae (19) by the difference (89) found in column **D**, to get the amount that we are to annex to the number 4864. This gives us the figures 214 to be annexed to our number, as shown in Step 7. We annex these three figures to the other four and we now have seven figures for our number.

To point off this number properly, we use the rule regarding the finding of the decimal point when given a positive characteristic. We have the characteristic of 5 in the logarithm of the product of the numbers shown in Step 3. To locate the decimal point, we begin at the left-hand figure and count off 5+1 figures, then place the decimal point as shown in Step 8. (See Table V.)

If you wish to check this problem to see whether it is right or not, you can multiply 2000 by 200, which are the round figures involved, and you will find there are six figures to the left of the decimal point. This proves that the problem is solved properly.

Now if you are to multiply the two numbers out in longhand to see whether it is correct or not, you will find that there is a slight difference in the decimal figures only. In other words your exact number would end in .8 instead of .4. You see your result is almost exactly what we obtained by the use of logs. Also you will notice that you have done very little actual figuring to get this product by logs.

In order to multiply 3, 4, 5, or 6 numbers together, all that is necessary is to add the logarithms of those numbers together, and finish the problem in the same way as Problem 1. Thus you see you can multiply 5 or 6 numbers together by the use of logarithms almost as easily as you can two. You would, of course, have the

extra logarithms to look up, otherwise the problem would be just the same as one operation for multiplication.

To show how easy it is to multiply several numbers together by logs, we will work out two problems in detail.

Problem 2. Multiply: $37 \times .46 \times .053 \times 7.9 \times 940$.

LOGARITHMIC TABLES

Page	N	0	Characteristic	Table
7	370	.568202	$2 - 1 = 1$	I
9	460	.662758	$0 + 1 = \bar{1}$	II
10	530	.724276	$1 + 1 = \bar{2}$	II
15	790	.897627	$1 - 1 = 0$	I
18	940	.973128	$3 - 1 = 2$	I

Log of 37. 1.568202

Log of .46 $\bar{1}.662758$

Log of .053 $\bar{2}.724276$

Log of 7.9 0.897627

Log of 940. 2.973128

Add the mantissae column by column and hold the number to be carried over from the last column.

Log of product = 3.825991 From the sum of the positive characteristics, plus what is carried over from adding the mantissae, subtract the sum of the minus characteristics which $= +3 + 3 - 3 = +3$ characteristic.

Mant. of 6698 = $.825945$ Mantissa for first 4 figures of answer.

Difference of mantissae 46

LOGARITHMIC TABLES

Page	N	8	D
13	669	.825945	65

46÷65=71− (71−) annexed to 6698=669871− Characteristic 3+1=4 Point off four figures, starting at the left, and it gives 6698.71 Answer

Problem 3. Multiply 250.9×36.72×.9875×.04756

LOGARITHMIC TABLES

Page	N	0	2	5	6	9	D
4	250	....	....	....	....	.399501	......
7	367	....	.564903	....	....	....	......
19	987	....	....	.994537	....	....	......
9	475	....	....	....	.677242	....	......
8	432	....	....	....	636087	....	100

Log of 250.9 = 2.399501
Log of 36.72 = 1.564903 Add logs for product
Log of .9875 = $\bar{1}$.994537
Log of .04756 = $\bar{2}$.677242

Characteristic	Table
3 − 1 = 2	I
2 − 1 = 1	I
0 + 1 = $\bar{1}$	II
1 + 1 = $\bar{2}$	II

Log of product = 2.636183 +2+3−3=2 characteristic (like Problem 2).

Mant. of 4326 = .636087 =Mantissa for first 4 figures of answer.

Difference = 96

96÷100=96 96 annexed to 4326=432696 Characteristic 2+1=3

Point off three figures, starting at the left, and it gives 432.696 Answer.

These problems illustrate the small amount of figuring required to solve them by logs when compared to arithmetic.

In our previous problems in multiplication we had only four figures in each number. Now we will illustrate a problem in which there are six figures in each number. This problem will be worked out in detail in order that you may be able to follow it all the way through.

Problem 4. Multiply 369.875 by 4863.45

Instruction	*Operation*
Step 1	**Step 1**
Find log of first number in problem. Do this by finding mantissa of first four figures of number; then find amount to be added for the other two figures. Next find the characteristic. Combine to get log of number.	Find log of 369.875 Mantissa of 3698 = .567967 Part added for 75 = 885 Characteristic = 2. Log of 369.875 = 2.5680555 .75×118 = 88.5 = added part 3 − 1 = 2 characteristic (Table I)

LOGARITHMIC TABLES

Page	N	8	9	D
7	369	567967	568084	118

Step 2	**Step 2**
Find log of second number in problem. Do this by finding mantissa of first four figures of number; then find amount to be added for the other two figures. Next find the characteristic. Combine to get log of number.	Find log of 4863.45 Mantissa of 4863 = .686904 Part added for 45 = 40 Characteristic = 3. Log of 4863.45 = 3.686944 .45×89 = 40.+ = added part 4 − 1 = 3 Characteristic (Table I)

LOGARITHMIC TABLES

Page	N	3	4	D
9	486	.686904	.686994	89

Step 3	**Step 3**
To multiply numbers, add the logs of those numbers together.	Add logs of 369.875 and 4863.45 Step 1 gives log 2.5680555 Step 2 gives log 3.686944 Log of product = 6.2549995

Step 4

Find number that corresponds to this logarithm.
Do this by finding the number in the Logarithmic Tables that has a mantissa the same as the mantissa of the product or slightly less.
Subtract the mantissae.

Step 4

Find the number that corresponds to mantissa of product

Step 3 mantissa	=	.2549995
Mantissa for 1798	=	.254790
Difference	=	2095
From column N	=	179
From column 8	=	8
First four figures	=	1798

LOGARITHMIC TABLES

Page	N	8	9	D
3	179	.254790	255031	242

Step 5

Find difference between mantissa in same horizontal line in the Logarithmic Tables, or use column D

Step 5

Mantissa of 1799	=	.255031
Mantissa of 1798	=	.254790
		241
Column D	=	242

Step 6

Find part to annex
Divide difference in Step 4 by difference in Step 5.

Step 6

Divide 2095 by 242

```
242 ( 2095 ) 865+
      1936
       1590
       1452
        1380
        1210
         170
```

Step 7

Place the quotient at right-hand end of the number found from the Logarithmic Tables in Step 4.

Step 7

Number in Step 4	=1798
Quotient in Step 6	865+
	1798865+

Step 8

Locate decimal point.
Number of figures to left of decimal equals characteristic plus one.
Begin at left and count seven figures, then place point

Step 8

Characteristic of log of product = 6

6 + 1 = 7 (Table V)
Place decimal point at right end of the number 1798865.+
The product of 369.875 × 4863.45 = 1798865.+ Answer

Below is shown the solution without explanation.

Step 1	Mant. of 3698	=	.567967
	Mant. of 75	=	885
	Log of 369.875	=	2.5680555
Step 2	Mant. of 4863	=	686904
	Mant. of 45	=	40
	Log of 4863.45	=	3.686944
Step 3	Log of 369.875	=	2.5680555
	Log of product	=	6.2549995
Step 4	Mant. of 1798	=	.254790
	Difference	=	2095
Step 5	Column **D**	=	242
Step 6	2095÷242=865+		
Step 7	865 annexed to 1798=1798865+		
	1798865.+ Answer		

Description of Solution. We first find the logarithms of the numbers. We find the mantissa of the first four figures of the first number by turning to page 7, as illustrated in Step 1. Under column **8** we find the mantissa. We have now the two figures 7 and 5 to account for. The 75 is 75/100 of 1 unit in the column for figure 8, the fourth figure of our number 3698. In other words, if we would add 25 to 75 we would have 100 or 1 to add to 8. Thus, to get the amount which we are to add to this mantissa we have 75/100 of the difference between the mantissae in columns **8** and **9**. Column **D** gives the difference 118, so we multiply 118 by .75, and point off accordingly, which gives 88.5 This is added to our mantissa, which gives the total mantissa for the whole number 369.875. We find the characteristic as shown in Table I.

We will now find the mantissa for the first four figures of the next number on page 9 of the Logarithmic Tables, as illustrated in Step 2. Under column **3** we find the proper mantissa. As in the first case, we have two figures left to find the amount we are to add on. This is found in the same way as before. This is added on in the same way and we find the characteristic in the same way as in Step 1.

Having our logarithms of the two numbers, we add them together in Step 3 for the log of the product, as we did in Problem 1. We have to find out what the corresponding number is for this logarithm as before. We will find the first four figures on page 3 of Logarithmic Tables as illustrated in Step 4. In column 8 we find the next less mantissa for the first four figures. This mantissa we subtract from the logarithm of the product and have a difference of 2095. This difference divided by the value in column D in Step 4 gives the value 865 (Step 6), which is the amount we annex to the four figures previously found (Step 7).

Next we place the decimal point in the right place by Table V. To the characteristic of 6 we add 1, which makes seven figures to the left of the decimal point. (Step 8.) This is the answer for the product of our two numbers.

While we have gone into quite a bit of detail to describe the operation to you, there is little actual figuring. When you compare this with the amount of time that would be required to multiply this out in the ordinary way, you will agree with us that logs can save you a lot of time besides eliminating the chances of errors.

Six-place tables are only accurate to sixth figure of answer.

Practice Problems

In order to give you practice in using the Logarithmic Tables, several practice problems in multiplication are given for you to solve by means of logarithms.

Follow the methods illustrated in the four problems already worked out. It is not necessary to send in the solutions of these practice problems unless you fail to get the answers given here. If you send them in, show all your work.

1. Multiply $820 \times 40 \times 293$	9610422. Answer
2. Multiply $423.1 \times 12.34 \times 65$	339367.9 Answer
3. Multiply $9.99 \times .777 \times 67.8$	526.278 Answer
4. Multiply $68.52 \times .0019 \times 371.8$	48.4039 Answer
5. Multiply 254.76×4.3219	1101.05 Answer
6. Multiply $.007342 \times .09837$	.000722233 Answer
7. Multiply $2700000 \times 39000000 \times 980000$	103,194,000,000,000,-000,000 Answer
8. Multiply 8100.07×3002.58	24,321,112 + Answer

DIVISION

In order to divide two numbers by the use of logarithms, we find the logarithms of the two numbers and then subtract these two logarithms instead of adding them as we did in multiplication. To illustrate this, we will use the same two numbers we did in Problem 1 and divide them.

Problem 5. Divide 220.2 by 2209

Instruction	*Operation*
Step 1	**Step 1**
Find log of the first number in problem. Do this by finding the mantissa of the number. Next find the characteristic. Combine the two and you have the log of the number.	Find log of 220.2 Mantissa of 220.2 = .342817 Characteristic of 220.2 = 2. Log of 220.2 = 2.342817 (See Table I)
Step 2	**Step 2**
Find log of the second number. Do this by finding the mantissa of the number. Next find the characteristic. Combine the two and you have the log of the number.	Find log of 2209 Mantissa of 2209 = .344196 Characteristic of 2209 = 3. Log of 2209 = 3.344196
Step 3	**Step 3**
To divide one number by another subtract the logs of the numbers. After 1 is borrowed from 2 to make 13 in the top line, instead of changing the 2 to 1, add 1 to the 3 in the bottom line, subtract the smaller number from the larger, and place a − sign in front of the difference: $+2-(3+1)=-2$ or $\bar{2}$	Subtract log of 2209 from log of 220.2 From Step 1 log of 220.2 = 2.342817 From Step 2 log of 2209 = 3.344196 Log of quotient = $\bar{2}$.998621

Finding the Number that Corresponds to the Logarithm

Step 4	**Step 4**
Find the first four figures of the number that corresponds to this logarithm. Do this by finding the number in Logarithmic Tables that has a mantissa the same as the mantissa in your problem, or slightly less. Subtract the mantissae.	Find number that corresponds to mantissa. From Step 3 log = 2.998621 Mantissa of 9968 = .998608 Difference = 13

LOGARITHMIC TABLES

Page	N	8	9	D
19	996	998608	998652	44

Figure 8 from Column 8 annexed to 996=9968.

Step 5
Find difference between mantissa in adjacent columns on same horizontal line in the Logarithmic Tables or use Column D.

Step 5

Mantissa of 9969	=	.998652
Mantissa of 9968	=	.998608
Difference	=	44
Column D	=	44

Step 6
Divide difference in Step 4 by difference in Step 5.

Step 6
Divide 13 by 44

```
44 ( 13.0 ) .29+
      8 8
      4 20
      3 96
```

Step 7
Place the quotient at the right-hand end of the number found from the Table, Step 4.

Step 7

Number in Step 4	=	9968
Quotient in Step 6	=	29
		996829

Step 8
Locate decimal point.
Number of ciphers to right of decimal point is negative characteristic minus one, or -2 less $1=-1$

Step 8
Characteristic of log is -2 less 1 $=-1$
Decimal point is to left of one cipher=.0996829
Thus 220.2÷2209=.0996829 Ans.

Below we show how simple the problem is when the explanation is omitted.

Step 1	Log of 220.2	=	2.342817
Step 2	Log of 2209	=	3.344196
Step 3	Log of quotient	=	$\bar{2}.998621$
Step 4	Mant. of 9968	=	.998608
	Difference	=	13
Step 5	Difference	=	44
Step 6	13÷44=29+		
Step 7	29 annexed to 9968=996829		
Step 8	.0996829 Answer		

Since we are dividing one number by a larger one our result, of course, will be a decimal number. In that case, we will have a negative characteristic as described in Table II.

You notice that the larger characteristic is subtracted from the smaller one. In order to do this we need to add two to the smaller characteristic. Thus, we will have as a result a minus 2 as a characteristic for the difference.

We now proceed the same as we did with the problem in multiplication to find the corresponding number. We find the next mantissa nearest to the one of the quotient as shown in Step 4. Subtract these two mantissae and you have a difference of 13. Now turn over to column **D** where we find the difference 44, Step 5. We divide 13 by 44 as we did in our multiplication problem and we have the result 29. This is annexed to your number as indicated by the location of your mantissa in your table and we have the result in Step 7.

In order to point this off properly, refer to Table V. Here we have a negative characteristic of 2, and we subtract 1, which is just the reverse of finding the characteristic. That gives us one cipher to the right of the decimal, so point it off as in Step 8.

If you want to check this result, you can divide by the ordinary way and see how near our log method is correct. In actual figures, however, we have done very little mathematical work. It takes quite a little time to describe this log method to you, but after you learn it once, you will find it comes easy.

Practice Problems

Solve the following problems by logarithms. Do not send in the solutions unless you fail to get the answers given. If you send them in, show all your work.

1. Divide 847.6 by 24.99	33.9176	Answer
2. Divide 7256.2 by 879.26	8.2526	Answer
3. Divide 276.543 by 912.34	.303113	Answer
4. Divide 0.9783 by .1234	7.92787	Answer
5. Divide .00879 by .0092	.955435	Answer
6. Divide 8321000 by 2135000	3.89743	Answer
7. Divide 850.06 by 50.082	16.9733	Answer
8. Divide .00097 by .0023	.421739	Answer

You will find that the knowledge of logarithms will come in very handy when you take up your advanced studies. In these texts there are a number of problems in which four, five, or six numbers are multiplied together and sometimes one or two divided. Both of these operations can be done at the same time by logarithms. In other words, you can multiply and divide at the same time. We will illustrate by a problem. Let us use three of the numbers, of which we already have the logarithms in our first table.

MULTIPLICATION AND DIVISION

Problem 6. Find the result $\frac{220.2 \times 22.01}{2209}$.

Step 1

$$\left.\begin{array}{ll} \text{Log of } 220.2 & = 2.342817 \\ \text{Log of } 22.01 & = \underline{1.342620} \end{array}\right\} \text{Add logs for product of number}$$

$$\left.\begin{array}{ll} \text{Log of product} & = 3.685437 \\ \text{Log of } 2209 & = \underline{3.344196} \end{array}\right\} \text{Subtract logs for quotient of numbers}$$

Log of quotient = 0.341241
Mant. of 2194,
Step 2 = .341237
Difference

Step 2

LOGARITHMIC TABLES

Page	N	4	5	D
4	219	.341237	.341435	198

Step 3

$4 \div 198 = 02$
Annexing 02 to 2194=219402

Step 4

Characteristic $0+1=1$ Point off one figure to left of decimal (Table V) 2.19402 Answer

In this problem, add the first two logarithms. This gives the logarithm of the product. Subtract the logarithm of the third num-

ber from the logarithm of the product. This gives the quotient. Now all we have to do is to find the corresponding number for this logarithm. In the Logarithmic Tables, as shown in Step 2, you will find the next mantissa nearest to the one we have for the quotient. The difference is 4. We divide the difference by the difference in column D, Step 2, which is 198. The result is 02. We annex 02 to the right-hand end of our number 2194 and we get 219402, which is the whole number we are looking for. To find where to put the decimal point, we refer to Table V. Having a zero characteristic, we add one and this gives us one figure to the left of the decimal.

Thus you see we finished these two operations almost as quickly as we did the one before. The only difference is the extra step of subtracting one of the logarithms from the sum of the other two. Thus you see the use of logarithms will save you quite a bit of time from what is necessary to work this kind of a problem by arithmetic.

Practice Problems

Solve the following problems by logarithms. Do not send in the solutions unless you fail to get the answers given. If you send them in, show all your work.

1. $\dfrac{67432 \times .02035}{700.09}$ 1.96009 Answer

2. $\dfrac{25.729 \times 1.0025}{35.678}$ .722947 Answer

3. $\dfrac{29000 \times 670000 \times 5300}{80029}$ 1,286,770,000 Answer

ROOTS AND POWERS

In the introductory discussion and in the lesson on Powers and Roots we explained the ordinary use of the square root or radical sign and exponents. This same sign is often used for other roots beside the square root. For instance, $\sqrt[3]{7235}$ means that the cube root of the number under the radical sign is to be obtained. We may also use any other figure in place of the 3 to indicate any root of the number under the radical sign. $\sqrt[4]{256}$ indicates that the fourth

root of 256 is to be obtained. The result equals 4, for $4\times4\times4\times4=256$. Also, $\sqrt[5]{243}$ indicates the fifth root of 243, which equals 3. Or $\sqrt[6]{64}$ indicates the sixth root of 64, which equals 2.

Ordinarily, there is a single figure used as an exponent for a number. The exponent indicates the power to which the number is to be increased. These exponents can be either fractional or decimal. $853^{2/3}$ is an illustration of a fractional exponent. It is read 853 to the 2/3rds power. This indicates that we are to raise 853 to the second power and extract the cube root. Thus, the numerator of the exponent indicates the power and the denominator indicates the root. $275^{7/5}$ is read 275 to the 7/5ths power. In this exponent the numerator is larger than the denominator. Thus it is seen that any kind of a fraction may be used as an exponent.

$724^{1.6}$ is an illustration of a decimal exponent. It is read 724 to the one and six-tenths power. This type of exponent may be either part or wholly decimal.

We have a great variety of exponents, but we can solve all problems containing these indicated operations by logarithms. Some of these operations are impossible by arithmetic. Therefore, you see how valuable logarithms are in this particular work.

In order to solve a problem in which we want to find the root, we find the logarithm of the number and then divide the logarithm by the exponent of the root of the number. For instance, if we want the square root of 75, we find the logarithm of 75 and divide the log by 2, which gives us the logarithm of the square root of the number. To find the cube root, we divide by 3; to find the fourth root, we divide by 4, etc.

To find the power of a number, we find the logarithm of the number and multiply it by the exponent of the power to which we want to raise the number. This gives the log of the power. You will note that this is just the reverse of finding the root.

For instance, if we want to find the square of 229, we find the logarithm of 229, and then multiply it by 2. That gives us the logarithm of the square of the number. If we multiply the logarithm of any given number by any indicated power, we obtain the logarithm of that power for that number. Fractional or decimal exponents are handled in a similar manner. We will work out two problems to illustrate how this work is done.

Problem 7. Solve $99^{9/7}$

Step 1

Log of 99	=	1.995635	—Logarithmic Tables, page 19
		9	Multiply by 9 to obtain the log of 9th power of number
		7)17.960715	Divide by 7 to obtain the log of the 7th root
Log of $99^{9/7}$		2.5658164+	
Mant. of 3679		.565730	
Difference		864	

Step 2

LOGARITHMIC TABLES

Page	N	9	D
7	367	565730	118

Step 3

864÷118=73

73 annexed to 3679=367973

Step 4

Characteristic 2+1=3

Point off three figures to the left of the decimal point.

367.973 Answer

First, find the mantissa for 99. This is easily located on Page 19 of Logarithmic Tables. The characteristic is located by Table I.

To get the ninth power of this logarithm, you multiply it by 9; and to get the seventh root of that, you divide it by 7. That gives you the logarithm of the two operations. We next find the corresponding number to this log as we have done in the other problems.

In this problem you will see you have saved yourself a lot of time in working it out with logs. You can find the ninth power by arithmetic, but you could not find the seventh root that way.

Problem 8. Solve $741.4^{1.6}$

Step 1

Log of 741.4	=	2.870053	Log Table, Page 14. Characteristic, Table I.
Multiply 1.6	=	1.6	Multiply by exponent to get the log of the power.
		17220318	
		2870053	
Log of power	=	4.5920848	
Mant. of 3909	=	592066	See Step 2
Difference	=	188	

Step 2

LOGARITHMIC TABLES

Page	N	9	D
7	390	.592066	111

Step 3

$188 \div 111 = 17-$

17− annexed to $3909 = 390917-$

Step 4

Characteristic $4+1=5$

Point off five figures to left of decimal point, Table V.

39091.7 Answer

We find the logarithm of 741.4 in the same way as we did in the other problems, then multiply by 1.6. That gives us the logarithm of the 741.4 to the 1.6 power. We find the nearest less mantissa in the Logarithmic Tables and subtract it as shown. Steps 3 and 4 are the same as in the other problems. This problem cannot be solved by arithmetic.

Practice Problems

Work out the following problems by logarithms. Do not send in the solutions unless you fail to get the given answers. If you send them in, show all of your work.

1. Find the value of $\sqrt[5]{476.92}$ — 3.43312 Answer
2. Find the value of $12345.^{9/3}$ — 1,881,365,200,000+ Answer
3. Find the value of $9.645^{3.5}$ — 2786.48 Answer

DECIMAL NUMBERS WITH EXPONENTS

Decimal numbers have negative characteristics, as we have shown in previous discussions. When we find it necessary to solve problems having decimal numbers with **powers** or **roots,** it will be necessary to change the negative characteristic to a positive characteristic.

As an example let us find the log of $.07854^3$

Step 1

Log of $.07854^3 = \bar{2}.895091 \times 3$ Table II

In order to avoid negative characteristics, we can make them positive by adding 10 to the characteristics and subtracting 10 at the end of the logarithms. This is illustrated in Step 2.

Step 2

$$\begin{array}{l} \bar{2}.895091 \times 3 \\ +10 \qquad\quad -10 \\ \hline (8.895091-10) \times 3 \end{array}$$

Multiplying as indicated, gives us Step 3. We can simplify, by **subtracting** 20 from each part of the characteristic as shown.

Step 3

$$\begin{array}{rr} 26.685273-30 & \qquad 26-30=\bar{4} \\ \underline{20 \qquad\quad 20} & \\ 6.685273-10 & \qquad +6-10=4 \end{array}$$

Now we can go back to the negative form by subtracting the 10 from the 6, which gives us a minus 4 characteristic with the mantissa shown in Step 4.

Step 4 $\bar{4}.685273$ Answer

This is the correct result, which cannot be obtained by multiplying directly the negative characteristics shown in Step 1.

Still another complication arises when you have the root instead of the power of a decimal number.

As an example let us find the log of $\sqrt[4]{.07854}$

This is the same number as used in the above problem, so we have the same logarithm and the negative characteristic, Steps 1 to 4.

Step 5 $(\bar{2}.895091 \div 4)$

In this case we have to divide by 4 in order to get the log of the root. Change the negative to the positive characteristic and we have Step 6.

Step 6 $(8.895091-10)\div 4$

In dividing such a logarithm we must have a -10 at the end after the division. Therefore, we must add to both parts of the characteristic to allow for this condition. In this case we add 30 as shown in Step 7. Dividing as indicated we get Step 8.

Step 7

$$\begin{array}{r} (8.895091-10)\div 4 \\ 30 \qquad\quad -30 \\ \hline 4)\overline{38.895091-40} \end{array}$$

Step 8 $9.72377275-10$

Now we can put this back in a negative form as before, and $9-10$ gives us -1 for a characteristic, as shown in Step 9. This is the correct logarithm for our problem.

Step 9 $\bar{1}.72377275$ Answer

This is the only method by which problems of this kind can be solved. We will solve a problem of this type to show you how it is done.

Problem 9. Solve: $\dfrac{(.07345)^3}{(.62547)^{1/2}}$

Log of $.07345=\bar{2}.865992$ Logarithmic Tables, Page 14. Characteristic, Table II.

Change to positive characteristic	$=8.865992-10$	$(8-10=-2)$
Multiply by 3 for log of power	$=\qquad 3$	
This gives log of numerator	$=26.597976-30$	
Mant. of 6254	$=.796158$	Logarithmic Tables, Page 12
Add for fifth figure 7	$=48.3$	$(7\times 69=48.3)$
Log of .62547	$=\bar{1}.796206.3$	Characteristic, Table II
Change to + characteristic	$=9.7962063-10$	$(9-10=-1)$
Add 10 to both parts	$=10 \qquad -10$	
Divide by 2 for root	$=2)\overline{19.7962063-20}$	
This gives log of denominator	$=9.89810315-10$	

Subtract log of denominator from log of numerator in order to divide the numbers.

Log of numerator $=26.597976-30$
Log of denominator $=\underline{\ 9.89810315-10}$
Log of quotient $=16.69987285-20$
Change to negative characteristic $=\bar{4}.69987285-(+16-20=-4)$
Mant. of 5010 $=\underline{\ .699838}$
Difference $=\quad 3485$

LOGARITHMIC TABLES

Page	N	0	D
10	501	699838	87

Divide problem difference by column D value. $34.85\div87=4+$

Annex to other figures at right-hand end. Annexing 4 to $5010=50104$

Table V, $4-1=3$

There will be three ciphers to right of decimal point.

.00050104 Answer

COMBINATION OF OPERATIONS

We are going to work out in detail a problem, which will illustrate four or five of the different operations at once. It is only a little different from the first problem illustrated, but is more complicated. The two numbers in the numerator both have simple exponents, while one number in the denominator has a fractional exponent and the other has a decimal exponent. And the radical is for the cube root of the total result of the other operations.

We have worked out this problem, giving the details of each operation so that you can follow it all the way through. Study this very carefully as it illustrates many of the different operations at once. It also shows you how much time and labor can be saved by using logarithms and how the chance for many mistakes is eliminated. Some of these operations cannot be solved by arithmetic.

After you have studied this problem through thoroughly so

that you understand it, you should be able to handle any kind of a problem which involves logarithms.

You must remember that you cannot add or subtract numbers after you get the logarithms of them. You must add or subtract by the ordinary methods before you get the logs. When you add the logarithms, you multiply the numbers; and when you subtract the logarithms, you divide the numbers.

Problem 10. Find value of: $\sqrt[3]{\dfrac{3678^4 \times 3.257^2}{1679^{2/3} \times 1.345^{1.5}}}$

Step 1

Find log of first number of denominator.
Do this by finding mantissa of number in Logarithmic Tables, page 3.
Next find characteristic of number and combine.
Next find log for power of number.
Next find log for root of number.

Step 1

Find log of $1679^{2/3}$

Mantissa of 1679	=	.225051
Characteristic, Table I	=	3.
Log of 1679	=	3.225051
Multiply by power		2
Divide by root		3)6.450102
Log of $1679^{2/3}$	=	2.150034

Step 2

Find log of second number of denominator.
Do this by finding mantissa of number in Logarithmic Tables, page 2.
Next find characteristic of number and combine.
Next find log for power of number.

Step 2

Find log of $1.345^{1.5}$

Mantissa of 1.345	=	.128722
Characteristic, Table I	=	0
Log of 1.345	=	0.128722
Multiply by power		1.5
		643610
		128722
Log of $1.345^{1.5}$	=	0.193083

Step 3

Find log of denominator.
Do this by adding the two logs of the numbers for log of product of numbers.

Step 3

Add logs of $1679^{2/3}$ and $1.345^{1.5}$

Step 1 gives log		2.150034
Step 2 gives log		0.193083
Log of denominator	=	2.343117

Step 4

Find log of first number of numerator.
Do this by finding mantissa of number in Logarithmic Tables, page 7.
Next find characteristic of number and combine.
Next find log for power of number.

Step 4

Find log of 3678^4

Mantissa of 3678	=	.565612
Characteristic, Table I	=	3.
Log of 3678	=	3.565612
Multiply by power		4
Log of 3678^4	=	14.262448

Step 5

Find log of second number of numerator.
Do this by finding mantissa of number in Logarithmic Tables, page 6. Next find characteristic of number and combine.
Next find log for power of number.

Step 5

Find log of 3257^2

Mantissa of 3257	=	.512818
Characteristic, Table I	=	0.
Log of 3257	=	0.512818
Multiply by power		2
Log of 3257^2	=	1.025636

Step 6

Find log of numerator.
Do this by adding the logs of the two numbers for log of product.

Step 6

Add logs of 3678^4 and 3.257^2

Step 4, log of 3678^4	=	14.262448
Step 5, log of 3.257^2	=	1.025636
Log of numerator	=	15.288084

Step 7

Find log of all numbers under the radical sign.
Do this by subtracting log of denominator from log of numerator for quotient of numerator divided by denominator.

Step 7

Subtract log of denominator from log of numerator

Step 6, log of numerator	=	15.288084
Step 3, log of denominator	=	2.343117
Log of quotient	=	12.944967

Step 8

Find root of combined results under radical sign.
Do this by dividing log of quotient of combined numerator and denominator by indicated root of the radical sign.

Step 8

Find cube root of number whose log

by Step 7	=	12.944967
Divide by 3		3)12.944967
Log of cube root	=	4.314989

Step 9

Find number that corresponds to log of Step 8.
Do this by finding the number in Logarithmic Tables that has a mantissa the same or a little less than mantissa of the log.
Subtract mantissa.

Step 9

Find number that corresponds to

mantissa of Step 8	=	.314989
Column N gives 206		
Column 5 gives 5		
2065		
Mantissa for 2065	=	.314920
Difference		69

LOGARITHMIC TABLES

Page	N	5	D
4	206	314920	210

Step 10

Find amount to be annexed to the number from the difference of mantissa of Step 9.
Do this by dividing this difference by the difference shown in column **D.**

Step 10

Divide 69 by 210

```
210 ( 69.0 ) .328
      63 0
       6 00
       4 20
       1 800
       1 680
```

Step 11

Place quotient of Step 10 at right end of number found in Step 9 to get all figures of resulting number.

Step 11

Number in Step 9	2065
Quotient in Step 10	328
	2065328

Step 12

Locate decimal point.
Number of figures to left of decimal point equals characteristic plus one.

Step 12

Step 8 Characteristic =	4
Add one	1
	5

Count five figures from left end and place decimal point.

20653.28 Answer

TRIAL EXAMINATION

Directions. This trial examination is to be used as a test to see whether you are ready for the regular or Final Examination.

Do not send us this trial examination.

Work all of the following problems. *After* you work the problems, check your answers with the solutions shown on Page 38 (top folio).

If you miss more than two of the problems it means you should review the whole Section very carefully.

Do not try this trial examination until you have worked every practice problem in the Section.

Do not start the final examination until you have completed this trial examination.

1. $3 \times 27 \times (95-32)$
2. 273.46×5.34127
3. $852.3 \div 42.341$
4. $.1563 \div .256$
5. 3.546^3
6. $576^{1/3}$
7. $8.23^{1.2}$

$$\sqrt{\frac{152 \times 27}{356 \div 94}}$$

EXAMINATION

Solve the following problems by the methods shown in Problems 2, 3, and 6 in the text. Show all details.

1. Solve by logs: $3 \times 27 \times (40.17 - 22.03)$
2. Solve by logs: 254.76×4.3219
3. Solve by logs: Divide 847.6 by $(612.7 - 22.5)$
4. Solve by logs: Divide .00879 by .0092
5. Find by logs the number for 3.456^5
6. Find by logs the number for $4789^{1/4}$
7. Find by logs the number for $.2798^{1.6}$
8. Find by logs the number for $789.2^{3/5}$
9. Find by logs the number for $20.04^{4/3}$
10. Solve the following problem, showing all details in the same way as shown in problem 10 in the text: $\sqrt{\dfrac{301 \times 192}{823 \div 25}}$

SOLUTIONS TO TRIAL EXAMINATION PROBLEMS

1. $3 \times 27 \times (95-32)$

Instruction	*Operation*
Step 1 Since subtraction cannot be performed by logarithms, perform the subtraction before taking logs.	**Step 1** $95-32=63$ Now the problem is $3 \times 27 \times 63$
Step 2 Find log of 3. To do this look up the mantissa of 3 in the table. Next, note that the characteristic of 3 is 0.	**Step 2** Mantissa of 3 $=0.477121$ Characteristic of 3 $=0.$ Log of 3 $=0.477121$
Step 3 Find log of 27. Do this by looking up the mantissa and combining it with the characteristic.	**Step 3** Mantissa of 27 $=.431364$ Characteristic of 27 $=1.$ Log of 27 $=1.431364$
Step 4 Find log of 63. Do this by looking up the mantissa of 63 and combining it with the characteristic.	**Step 4** Mantissa of 63 $=.799341$ Characteristic of 63 $=1.$ Log of 63 $=1.799341$
Step 5 To multiply numbers, add the logs of those numbers together. Thus add log of 3, log of 27 and log of 63.	**Step 5** From Step 2, log of 3 $=0.477121$ From Step 3, log of 27 $=1.431364$ From Step 4, log of 63 $=1.799341$ Log of product $=3.707826$
Step 6 Find number that corresponds to this logarithm. Do this by finding the number in Logarithmic Tables that has a mantissa the same as the mantissa in the problem, or the next *smaller* mantissa. In this case the exact mantissa is in the table.	**Step 6** Find number that corresponds to given mantissa. From Step 5, mantissa $=.707826$ Mantissa of 5103 $=.707826$ Result, 5103
Step 7 Locate decimal point. Number of figures to left of decimal point is characteristic $+1$. Begin at left of result in Step 6 and count four figures to right and place point.	**Step 7** Characteristic is 3. $3+1=4$ Decimal point is after *3*. *5103.* Answer.

2. 273.46×5.34127

Instruction / *Operation*

Step 1

Find log of first number in problem. Do this by finding mantissa of first four figures of number; then find amount to be added for other figure. Next find characteristic. Combine to get log of number.

Step 1

Find log of 273.46

Mantissa of 2734 =	.436799
Part added for 6 =	.000095
Characteristic =	2.
Log of 273.46 =	2.436894

$.6 \times .000159 = .000095 =$ added part
$3 - 1 = 2$ characteristic

Step 2

Find log of second number in problem. Do this by finding mantissa of first four figures of number; then find amount to be added for the other two figures. Next find the characteristic. Combine to get log of number.

Step 2

Find log of 5.34127

Mantissa of 5341 =	.727623
Part added for 27 =	.000022
Characteristic =	0.
Log of 5.34127 =	0.727645

$.27 \times .000081 = .000022 =$ added part
$1 - 1 = 0$ characteristic

Step 3

To multiply numbers, add the logs of those numbers. Thus add logs of 273.46 and 5.34127.

Step 3

Step 1 gives log	2.436894
Step 2 gives log	0.727645
Log of product	3.164539

Step 4

Find number that corresponds to this logarithm. Do this by finding the number in the Logarithmic Tables that has a mantissa the same as the mantissa of the product, or slightly less.

Step 4

Step 3 mantissa =	.164539
Mantissa of 1460 =	.164353
Difference =	.000186

From column N =	146
From column O =	0
First four figures =	1460

Step 5

Find difference between mantissa in same horizontal line in the Logarithmic Tables, or use Column D.

Step 5

Mantissa of 1461 =	.164650
Mantissa of 1460 =	.164353
Difference =	.000297

Column D = .000297

Step 6

Find part to annex. Divide difference in Step 4 by difference in Step 5.

Step 6

Divide .000186 by .000297
$.000186 \div .000297 \doteq .63$

Step 7 Place quotient at right-hand end of the number found from the Logarithmic Tables in Step 4.	**Step 7** Number in Step 4 = 1460; Quotient in Step 6 = 63; 146063
Step 8 Locate decimal point. Number of figures to left of decimal equals characteristic plus one. Begin at left and count four figures, then place point.	**Step 8** Characteristic of log of product = 3; $3+1=4$; Place point after *0*. The product of $273.46 \times 5.34127 = 1460.63$ Answer

3. $852.3 \div 42.341$

Instruction	*Operation*
Step 1 Find log of the first number in problem. Do this by finding the mantissa of the number. Next find the characteristic. Combine the two to get the log of the number.	**Step 1** Find log of 852.3 Mantissa of 852.3 = .930592 Characteristic of 852.3 = 2. Log of 852.3 = 2.930592
Step 2 Find log of the second number. Do this by finding mantissa of first four figures of number; then find amount to be added for the other figure. Next find the characteristic. Combine to get log of number.	**Step 2** Find log of 42.341 Mantissa of 4234 = .626751 Part added for 1 = .000010 Characteristic = 1. Log of 42.341 = 1.626761 $.1 \times .000103 = .000010$ = added part $2-1=1$ characteristic
Step 3 To divide one number by another, subtract the logs of the numbers. Thus subtract log of 42.341 from log of 852.3	**Step 3** From Step 1 log of 852.3 = 2.930592 From Step 2 log of 42.341 = 1.626761 Log of quotient = 1.303831

Finding the Number that Corresponds to the Logarithm

Step 4 Find the first four figures of the number that corresponds to this logarithm. Do this by finding a number in the Logarithmic Tables that has a mantissa the same as the mantissa in the problem, or slightly less. Subtract the mantissae.	**Step 4** Find number that corresponds to mantissa. From Step 3 mantissa = .303831 Mantissa of 2012 = .303628 Difference = .000203

Step 5

Find difference between mantissae in adjacent columns on same horizontal line in the Logarithmic Tables, or use Column D.

Step 5

Mantissa of 2013 = .303844
Mantissa of 2012 = .303628
Difference = .000216
Column D = .000216

Step 6

Divide difference in Step 4 by difference in Step 5.

Step 6

$.000203 \div .000216 = .94$

Step 7

Place the quotient at the right-hand end of the number found from the Table, Step 4.

Step 7

Number in Step 4 = 2012
Quotient in Step 6 = 94
201294

Step 8

Locate decimal point. Number of figures to left of decimal equals characteristic plus one. Begin at left and count two figures then place decimal point.

Step 8

Characteristic of log of quotient = 1

$1+1=2$

Place decimal point between *0* and *1*. Thus

$852.3 \div 42.341 = 20.1294$

4. $.1563 \div .256$

Instruction / *Operation*

Step 1

Find log of the first number in the problem. Do this by finding the mantissa of the number. Next find the characteristic. When there are no figures to left of decimal point, the characteristic is negative and one more than the number of ciphers to right of decimal point, or -1. This may be written as $9-10$ or $19-20$.

Step 1

Find log of .1563

Mantissa of 1563 = .193959
Characteristic of .1563 = $-1 = 19-20$
Log of .1563 = $19.193959-20$

Step 2

Find log of second number in the problem. Do this by finding the mantissa of the number. Next find the characteristic. Combine the two to get log of the number.

Step 2

Find log of .256

Mantissa of 256 = .408240
Characteristic of .256 = $-1 = 9-10$
Log of .256 = $9.408240-10$

Step 3

To divide one number by another subtract the logs of the numbers.

Step 3

Subtract log of .256 from log of .1563

From Step 1, log of .1563 = $19.193959-20$
From Step 2, log of .256 = $9.408240-10$
Log of quotient = $9.785719-10$

Step 4

Find the first four figures of the number that corresponds to this logarithm. Do this by finding the number in Logarithmic Tables that has a mantissa the same as the mantissa in the problem, or slightly less. Subtract the mantissae.

Step 4

Find number that corresponds to mantissa.

From Step 3 log	$= 9.785719 - 10$
Mantissa of 6105	$= .785686$
Difference	$= .000033$

Step 5

Find difference between mantissae in adjacent columns on same horizontal line in the Logarithmic Tables, or use Column D.

Step 5

Mantissa of 6106	$= .785757$
Mantissa of 6105	$= .785686$
Difference	$= .000071$

Step 6

Divide difference in Step 4 by difference in Step 5.

Step 6

$.000033 \div .000071 = .46$

Step 7

Place the quotient at the right-hand end of the number found from Table, in Step 4.

Step 7

Number in Step 4	$= 6105$
Quotient in Step 6	$= 46$
	610546

Step 8

Locate decimal point. Number of ciphers to right of decimal point is negative characteristic minus one, or $1 - 1 = 0$.

Step 8

Characteristic of log is $9 - 10$ or negative characteristic one,

$$1 - 1 = 0$$

Decimal has no ciphers after it.

Thus $.1563 \div .256 = .610546$ Answer.

5. Solve 3.546^3

Step 1

Log of 3.546	$= 0.549739$
	3
Log of 3.546^3	$= 1.649217$
Mantissa of 4458	$= .649140$
Difference	$= .000077$

Logarithmic Tables, p. 7. Multiply by 3 to obtain the 3rd power of the number.

Step 2

$.000077 \div .000097 = .80$

80 annexed to $4458 = 445880$

The total difference D is .000097.

Step 3

Characteristic 1,

$$1+1=2$$

Point off 2 figures starting at left.

44.588 Answer.

6. Solve $576^{1/3}$

Step 1

Log of 576 = 2.760422		Logarithmic Tables, p. 11. Multiply by $\frac{1}{3}$ or divide by 3 (which are the same) to find the cube root of 576.
	3)2.760422	
Log of $576^{1/3}$	= 0.920141	
Mantissa of 8320	= .920123	
Difference	= .000018	

Step 2

.000018 ÷ .000052 = .35 — The difference between the mantissae of 8321 and 8320 is .000052, see Column **D**.

35 annexed to 8320 = 832035

Step 3

Characteristic 0.

$$0+1=1$$

Point off one figure from left.

8.32035 Answer.

7. Solve $8.23^{1.2}$

Step 1

Log of 8.23	= 0.915400	Log Table, page 16. Multiply by exponent to get log of power.
Multiply by 1.2	= 1.2	
	1830800	
	915400	
Log of power	= 1.0984800	
Mantissa of 1254	= .098298	
Difference	= .000182	

Step 2

.000182 ÷ .000346 = .53 — The difference between mantissae of 1255 and 1254 is .000346.

53 annexed to 1254 = 125453

Step 3

Characteristic 1

$$1+1=2$$

Point off two figures beginning at the left and insert decimal point.

12.5453 Answer.

8. Find value of:

Instruction	*Operation*
Step 1 Find log of first number of denominator. Do this by finding mantissa of number in Logarithmic Tables, page 7. Next find characteristic of number and combine.	**Step 1** Find log of 356. Mantissa of 356 = .551450 Characteristic = 2. Log of 356 = 2.551450
Step 2 Find log of second number of denominator. Do this by finding mantissa of number in Logarithmic Tables, page 18. Next find characteristic of number and combine.	**Step 2** Find log of 94 Mantissa of 94 = .973128 Characteristic = 1. Log of 94 = 1.973128
Step 3 Find log of denominator. Do this by subtracting the logs to find the quotient of the two numbers.	**Step 3** Subtract log of 94 from log of 356. From Step 1, log of 356 = 2.551450 From Step 2, log of 94 = 1.973128 Log of denominator = 0.578322
Step 4 Find log of first number of numerator. Do this by finding mantissa of number in Logarithmic Tables, page 3. Next find characteristic of number and combine.	**Step 4** Find log of 152 Mantissa of 152 = .181844 Characteristic = 2. Log of 152 = 2.181844
Step 5 Find log of second number of numerator. Do this by finding mantissa of number in Logarithmic Tables, page 5. Next find characteristic of number and combine.	**Step 5** Find log of 27 Mantissa of 27 = .431364 Characteristic = 1. Log of 27 = 1.431364
Step 6 Find log of numerator. Do this by adding the logs of the two numbers for log of product.	**Step 6** Add logs of 152 and 27. Step 4, log of 152 = 2.181844 Step 5, log of 27 = 1.431364 Log of numerator = 3.613208

Step 7

Find log of all numbers under radical sign. Do this by subtracting log of denominator from log of numerator for quotient of numerator divided by denominator.

Step 7

Subtract log of denominator from log of numerator.

Step 6, log of numerator	=3.613208
Step 3, log of denominator	=0.578322
Log of quotient	=3.034886

Step 8

Find root of combined results under radical sign. Do this by dividing log of quotient of combined numerator and denominator by indicated root of the radical sign.

Step 8

Find square root of number whose log by Step 7 =3.034886.

Divide by 2

$$2\overline{)3.034886}$$

Log of square root = 1.517443

Step 9

Find number that corresponds to log of Step 8. Do this by finding the number in Logarithmic Tables that has a mantissa the same or slightly less than the mantissa of the log. Subtract mantissa.

Step 9

Find number that corresponds to mantissa of Step 8 =.517443

Column N gives 329

Column 1 gives 1.

Mantissa of 3291	=.517328
Difference	=.000115

Step 10

Find amount to be annexed to the number. Do this by dividing the difference of Step 9 by the difference shown in Column D.

Step 10

Divide .000115 by .000132.

.000115÷.000132=.87

Step 11

Place quotient of Step 10 at right end of number found in Step 9 to get all figures of resulting number.

Step 11

Number in Step 9	=3291
Quotient in Step 10 =	87
	329187

Step 12

Locate decimal point. Number of figures to left of decimal point equals characteristic plus one.

Step 12

Step 8, characteristic	1
Add One	1
	2

Count two figures from left end and place decimal point.

32.9187 Answer.

Appendix

Six-Place Logarithmic Table

From 1 to 9999

N.	0	1	2	3	4	5	6	7	8	9	D.
100	00 0000	00 0434	00 0868	00 1301	00 1734	00 2166	00 2598	00 3029	00 3461	00 3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	01 0300	01 0724	01 1147	01 1570	01 1993	01 2415	424
103	01 2837	01 3259	01 3680	01 4100	4521	4940	5360	5779	6197	6616	420
104	7033	7451	7868	8284	8700	9116	9532	9947	02 0361	02 0775	416
105	02 1189	02 1603	02 2016	02 2428	02 2841	02 3252	02 3664	02 4075	02 4486	02 4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	03 0195	03 0600	03 1004	03 1408	03 1812	03 2216	03 2619	03 3021	404
108	03 3424	03 3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	04 0207	04 0602	04 0998	397
110	04 1393	04 1787	04 2182	04 2576	04 2969	04 3362	04 3755	04 4148	04 4540	04 4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
112	9218	9606	9993	05 0380	05 0766	05 1153	05 1538	05 1924	05 2309	05 2694	386
113	05 3078	05 3463	05 3846	4230	4613	4996	5378	5760	6142	6524	383
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	06 0320	379
115	06 0698	06 1075	06 1452	06 1829	06 2206	06 2582	06 2958	06 3333	06 3709	06 4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
117	8186	8557	8928	9298	9668	07 0038	07 0407	07 0776	07 1145	07 1514	370
118	07 1882	07 2250	07 2617	07 2985	07 3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	07 9181	07 9543	07 9904	08 0266	08 0626	08 0987	08 1347	08 1707	08 2067	08 2426	360
121	08 2785	08 3144	08 3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	09 0258	09 0611	09 0963	09 1315	09 1667	09 2018	09 2370	09 2721	09 3071	352
124	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	09 6910	09 7257	09 7604	09 7951	09 8298	09 8644	09 8990	09 9335	09 9681	10 0026	346
126	10 0371	10 0715	10 1059	10 1403	10 1747	10 2091	10 2434	10 2777	10 3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	11 0253	338
129	11 0590	11 0926	11 1263	11 1599	11 1934	11 2270	11 2605	11 2940	11 3275	3609	335
130	11 3943	11 4277	11 4611	11 4944	11 5278	11 5611	11 5943	11 6276	11 6608	11 6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	12 0245	330
132	12 0574	12 0903	12 1231	12 1560	12 1888	12 2216	12 2544	12 2871	12 3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	13 0012	323
135	13 0334	13 0655	13 0977	13 1298	13 1619	13 1939	13 2260	13 2580	13 2900	13 3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
138	9879	14 0194	14 0508	14 0822	14 1136	14 1450	14 1763	14 2076	14 2389	14 2702	314
139	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	14 6128	14 6438	14 6748	14 7058	14 7367	14 7676	14 7985	14 8294	14 8603	14 8911	309
141	9219	9527	9835	15 0142	15 0449	15 0756	15 1063	15 1370	15 1676	15 1982	307
142	15 2288	15 2594	15 2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	16 0168	16 0469	16 0769	16 1068	301
145	16 1368	16 1667	16 1967	16 2266	16 2564	16 2863	16 3161	16 3460	16 3758	16 4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	17 0262	17 0555	17 0848	17 1141	17 1434	17 1726	17 2019	17 2311	17 2603	17 2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	17 6091	17 6381	17 6670	17 6959	17 7248	17 7536	17 7825	17 8113	17 8401	17 8689	289
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
150	17 6091	17 6381	17 6670	17 6959	17 7248	17 7536	17 7825	17 8113	17 8401	17 8689	289
151	8977	9264	9552	9839	18 0126	18 0413	18 0699	18 0986	18 1272	18 1558	287
152	18 1844	18 2129	18 2415	18 2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	19 0051	281
155	19 0332	19 0612	19 0892	19 1171	19 1451	19 1730	19 2010	19 2289	19 2567	19 2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	20 0029	20 0303	20 0577	20 0850	20 1124	274
159	20 1397	20 1670	20 1943	20 2216	20 2488	2761	3033	3305	3577	3848	272
160	20 4120	20 4391	20 4663	20 4934	20 5204	20 5475	20 5746	20 6016	20 6286	20 6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	21 0051	21 0319	21 0586	21 0853	21 1121	21 1388	21 1654	21 1921	267
163	21 2188	21 2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	21 7484	21 7747	21 8010	21 8273	21 8536	21 8798	21 9060	21 9323	21 9585	21 9846	262
166	22 0108	22 0370	22 0631	22 0892	22 1153	22 1414	22 1675	22 1936	22 2196	22 2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	23 0193	256
170	23 0449	23 0704	23 0960	23 1215	23 1470	23 1724	23 1979	23 2234	23 2488	23 2742	255
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	24 0050	24 0300	250
174	24 0549	24 0799	24 1048	24 1297	24 1546	24 1795	24 2044	24 2293	2541	2790	249
175	24 3038	24 3286	24 3534	24 3782	24 4030	24 4277	24 4525	24 4772	24 5019	24 5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	25 0176	245
178	25 0420	25 0664	25 0908	25 1151	25 1395	25 1638	25 1881	25 2125	25 2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	25 5273	25 5514	25 5755	25 5996	25 6237	25 6477	25 6718	25 6958	25 7198	25 7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	26 0071	26 0310	26 0548	26 0787	26 1025	26 1263	26 1501	26 1739	26 1976	26 2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	26 7172	26 7406	26 7641	26 7875	26 8110	26 8344	26 8578	26 8812	26 9046	26 9279	234
186	9513	9746	9980	27 0213	27 0446	27 0679	27 0912	27 1144	27 1377	27 1609	233
187	27 1842	27 2074	27 2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	27 8754	27 8982	27 9211	27 9439	27 9667	27 9895	28 0123	28 0351	28 0578	28 0806	228
191	28 1033	28 1261	28 1488	28 1715	28 1942	28 2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	29 0035	29 0257	29 0480	29 0702	29 0925	29 1147	29 1369	29 1591	29 1813	29 2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	30 0161	30 0378	30 0595	30 0813	218
200	30 1030	30 1247	30 1464	30 1681	30 1898	30 2114	30 2331	30 2547	30 2764	30 2980	217
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
200	30 1030	30 1247	30 1464	30 1681	30 1898	30 2114	30 2331	30 2547	30 2764	30 2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	31 0056	31 0268	31 0481	31 0693	31 0906	31 1118	31 1330	31 1542	212
205	31 1754	31 1966	31 2177	31 2389	31 2600	31 2812	31 3023	31 3234	31 3445	31 3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	32 0146	32 0354	32 0562	32 0769	32 0977	32 1184	32 1391	32 1598	32 1805	32 2012	207
210	32 2219	32 2426	32 2633	32 2839	32 3046	32 3252	32 3458	32 3665	32 3871	32 4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	33 0008	33 0211	203
214	33 0414	33 0617	33 0819	33 1022	33 1225	33 1427	33 1630	33 1832	2034	2236	202
215	33 2438	33 2640	33 2842	33 3044	33 3246	33 3447	33 3649	33 3850	33 4051	33 4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	34 0047	34 0246	199
219	34 0444	34 0642	34 0841	34 1039	34 1237	34 1435	34 1632	34 1830	2028	2225	198
220	34 2423	34 2620	34 2817	34 3014	34 3212	34 3409	34 3606	34 3802	34 3999	34 4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	35 0054	194
224	35 0248	35 0442	35 0636	35 0829	35 1023	35 1216	35 1410	35 1603	35 1796	1989	193
225	35 2183	35 2375	35 2568	35 2761	35 2954	35 3147	35 3339	35 3532	35 3724	35 3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	36 0025	36 0215	36 0404	36 0593	36 0783	36 0972	36 1161	36 1350	36 1539	189
230	36 1728	36 1917	36 2105	36 2294	36 2482	36 2671	36 2859	36 3048	36 3236	36 3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	37 0143	37 0328	37 0513	37 0698	37 0883	185
235	37 1068	37 1253	37 1437	37 1622	37 1806	37 1991	37 2175	37 2360	37 2544	37 2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	38 0030	181
240	38 0211	38 0392	38 0573	38 0754	38 0934	38 1115	38 1296	38 1476	38 1656	38 1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7924	8101	8279	8456	8634	8811	8989	178
245	38 9166	38 9343	38 9520	38 9698	38 9875	39 0051	39 0228	39 0405	39 0582	39 0759	177
246	39 0935	39 1112	39 1288	39 1464	39 1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	39 7940	39 8114	39 8287	39 8461	39 8634	39 8808	39 8981	39 9154	39 9328	39 9501	173
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D.**

N.	0	1	2	3	4	5	6	7	8	9	D.
250	39 7940	39 8114	39 8287	39 8461	39 8634	39 8808	39 8981	39 9154	39 9328	39 9501	173
251	9674	9847	40 0020	40 0192	40 0365	40 0538	40 0711	40 0883	40 1056	40 1228	173
252	40 1401	40 1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	40 6540	40 6710	40 6881	40 7051	40 7221	40 7391	40 7561	40 7731	40 7901	40 8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	41 0102	41 0271	41 0440	41 0609	41 0777	41 0946	41 1114	41 1283	41 1451	169
258	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	41 4973	41 5140	41 5307	41 5474	41 5641	41 5808	41 5974	41 6141	41 6308	41 6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	42 0121	42 0286	42 0451	42 0616	42 0781	42 0945	42 1110	42 1275	42 1439	165
264	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	42 3246	42 3410	42 3574	42 3737	42 3901	42 4065	42 4228	42 4392	42 4555	42 4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	43 0075	43 0236	43 0398	43 0559	43 0720	43 0881	43 1042	43 1203	161
270	43 1364	43 1525	43 1685	43 1846	43 2007	43 2167	43 2328	43 2488	43 2649	43 2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	43 9333	43 9491	43 9648	43 9806	43 9964	44 0122	44 0279	44 0437	44 0594	44 0752	158
276	44 0909	44 1066	44 1224	44 1381	44 1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	44 7158	44 7313	44 7468	44 7623	44 7778	44 7933	44 8088	44 8242	44 8397	44 8552	155
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293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
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295	46 9822	46 9969	47 0116	47 0263	47 0410	47 0557	47 0704	47 0851	47 0998	47 1145	147
296	47 1292	47 1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
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307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
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419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
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422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
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454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
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468	67 0246	67 0339	67 0431	67 0524	67 0617	67 0710	67 0802	67 0895	0988	1080	93
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475	67 6694	67 6785	67 6876	67 6968	67 7059	67 7151	67 7242	67 7333	67 7424	67 7516	91
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477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	9428	9519	9610	9700	9791	9882	9973	68 0063	68 0154	68 0245	91
479	68 0336	68 0426	68 0517	68 0607	68 0698	68 0789	68 0879	0970	1060	1151	91
480	68 1241	68 1332	68 1422	68 1513	68 1603	68 1693	68 1784	68 1874	68 1964	68 2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	68 5742	68 5831	68 5921	68 6010	68 6100	68 6189	68 6279	68 6368	68 6458	68 6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	9309	9398	9486	9575	9664	9753	9841	9930	69 0019	69 0107	89
490	69 0196	69 0285	69 0373	69 0462	69 0550	69 0639	69 0728	69 0816	69 0905	69 0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	69 4605	69 4693	69 4781	69 4868	69 4956	69 5044	69 5131	69 5219	69 5307	69 5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
500	69 8970	69 9057	69 9144	69 9231	69 9317	69 9404	69 9491	69 9578	69 9664	69 9751	87
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502	70 0704	70 0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	70 3291	70 3377	70 3463	70 3549	70 3635	70 3721	70 3807	70 3893	70 3979	70 4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	70 7570	70 7655	70 7740	70 7826	70 7911	70 7996	70 8081	70 8166	70 8251	70 8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	71 0033	85
513	71 0117	71 0202	71 0287	71 0371	71 0456	71 0540	71 0625	71 0710	71 0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	71 1807	71 1892	71 1976	71 2060	71 2144	71 2229	71 2313	71 2397	71 2481	71 2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
520	71 6003	71 6087	71 6170	71 6254	71 6337	71 6421	71 6504	71 6588	71 6671	71 6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	72 0077	83
525	72 0159	72 0242	72 0325	72 0407	72 0490	72 0573	72 0655	72 0738	72 0821	72 0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	83
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
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531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	72 8354	72 8435	72 8516	72 8597	72 8678	72 8759	72 8841	72 8922	72 9003	72 9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	73 0055	73 0136	73 0217	73 0298	73 0378	73 0459	73 0540	73 0621	73 0702	81
538	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
540	73 2394	73 2474	73 2555	73 2635	73 2715	73 2796	73 2876	73 2956	73 3037	73 3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	73 6397	73 6476	73 6556	73 6635	73 6715	73 6795	73 6874	73 6954	73 7034	73 7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
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550	74 0363	74 0442	74 0521	74 0600	74 0678	74 0757	74 0836	74 0915	74 0994	74 1073	79
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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	74 4293	74 4371	74 4449	74 4528	74 4606	74 4684	74 4762	74 4840	74 4919	74 4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	74 8188	74 8266	74 8343	74 8421	74 8498	74 8576	74 8653	74 8731	74 8808	74 8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	9736	9814	9891	9968	75 0045	75 0123	75 0200	75 0277	75 0354	75 0431	77
563	75 0508	75 0586	75 0663	75 0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	75 2048	75 2125	75 2202	75 2279	75 2356	75 2433	75 2509	75 2586	75 2663	75 2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	75 5875	75 5951	75 6027	75 6103	75 6180	75 6256	75 6332	75 6408	75 6484	75 6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	75 9668	75 9743	75 9819	75 9894	75 9970	76 0045	76 0121	76 0196	76 0272	76 0347	75
576	76 0422	76 0498	76 0573	76 0649	76 0724	0799	0875	0950	1025	1101	75
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
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586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
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588	9377	9451	9525	9599	9673	9746	9820	9894	9968	77 0042	74
589	77 0115	77 0189	77 0263	77 0336	77 0410	77 0484	77 0557	77 0631	77 0705	0778	74
590	77 0852	77 0926	77 0999	77 1073	77 1146	77 1220	77 1293	77 1367	77 1440	77 1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	77 4517	77 4590	77 4663	77 4736	77 4809	77 4882	77 4955	77 5028	77 5100	77 5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
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605	78 1755	78 1827	78 1899	78 1971	78 2042	78 2114	78 2186	78 2258	78 2329	78 2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
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611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
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615	78 8875	78 8946	78 9016	78 9087	78 9157	78 9228	78 9299	78 9369	78 9440	78 9510	71
616	9581	9651	9722	9792	9863	9933	79 0004	79 0074	79 0144	79 0215	70
617	79 0285	79 0356	79 0426	79 0496	79 0567	79 0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	79 2392	79 2462	79 2532	79 2602	79 2672	79 2742	79 2812	79 2882	79 2952	79 3022	70
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622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
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624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	79 5880	79 5949	79 6019	79 6088	79 6158	79 6227	79 6297	79 6366	79 6436	79 6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	79 9341	79 9409	79 9478	79 9547	79 9616	79 9685	79 9754	79 9823	79 9892	79 9961	69
631	80 0029	80 0098	80 0167	80 0236	80 0305	80 0373	80 0442	80 0511	80 0580	80 0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
635	80 2774	80 2842	80 2910	80 2979	80 3047	80 3116	80 3184	80 3252	80 3321	80 3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	80 6180	80 6248	80 6316	80 6384	80 6451	80 6519	80 6587	80 6655	80 6723	80 6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	80 9560	80 9627	80 9694	80 9762	80 9829	80 9896	80 9964	81 0031	81 0098	81 0165	67
646	81 0233	81 0300	81 0367	81 0434	81 0501	81 0569	81 0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	81 2913	81 2980	81 3047	81 3114	81 3181	81 3247	81 3314	81 3381	81 3448	81 3514	67
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651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	81 6241	81 6308	81 6374	81 6440	81 6506	81 6573	81 6639	81 6705	81 6771	81 6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	81 9544	81 9610	81 9676	81 9741	81 9807	81 9873	81 9939	82 0004	82 0070	32 0136	66
661	82 0201	82 0267	82 0333	82 0399	82 0464	82 0530	82 0595	0661	0727	0792	66
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663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	82 2822	82 2887	82 2952	82 3018	82 3083	82 3148	82 3213	82 3279	82 3344	82 3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	82 6075	82 6140	82 6204	82 6269	82 6334	82 6399	82 6464	82 6528	82 6593	82 6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	82 9304	82 9368	82 9432	82 9497	82 9561	82 9625	82 9690	82 9754	82 9818	82 9882	64
676	9947	83 0011	83 0075	83 0139	83 0204	83 0268	83 0332	83 0396	83 0460	83 0525	64
677	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	83 2509	83 2573	83 2637	83 2700	83 2764	83 2828	83 2892	83 2956	83 3020	83 3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	63
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	83 5691	83 5754	83 5817	83 5881	83 5944	83 6007	83 6071	83 6134	83 6197	83 6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	83 8849	83 8912	83 8975	83 9038	83 9101	83 9164	83 9227	83 9289	83 9352	83 9415	63
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	84 0043	63
692	84 0106	84 0169	84 0232	84 0294	84 0357	84 0420	84 0482	84 0545	84 0608	0671	63
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	62
695	84 1985	84 2047	84 2110	84 2172	84 2235	84 2297	84 2360	84 2422	84 2484	84 2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
700	84 5098	84 5160	84 5222	84 5284	84 5346	84 5408	84 5470	84 5532	84 5594	84 5656	62
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701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	84 8189	84 8251	84 8312	84 8374	84 8435	84 8497	84 8559	84 8620	84 8682	84 8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	85 0033	85 0095	85 0156	85 0217	85 0279	85 0340	85 0401	85 0462	85 0524	85 0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	85 1258	85 1320	85 1381	85 1442	85 1503	85 1564	85 1625	85 1686	85 1747	85 1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	85 4306	85 4367	85 4428	85 4488	85 4549	85 4610	85 4670	85 4731	85 4792	85 4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	61
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721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
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725	86 0338	86 0398	86 0458	86 0518	86 0578	86 0637	86 0697	86 0757	86 0817	86 0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	86 3323	86 3382	86 3442	86 3501	86 3561	86 3620	86 3680	86 3739	86 3799	86 3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	86 6287	86 6346	86 6405	86 6465	86 6524	86 6583	86 6642	86 6701	86 6760	86 6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
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741	9818	9877	9935	9994	87 0053	87 0111	87 0170	87 0228	87 0287	87 0345	59
742	87 0404	87 0462	87 0521	87 0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	87 2156	87 2215	87 2273	87 2331	87 2389	87 2448	87 2506	87 2564	87 2622	87 2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	87 5061	87 5119	87 5177	87 5235	87 5293	87 5351	87 5409	87 5466	87 5524	87 5582	58
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D.**

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751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	87 7947	87 8004	87 8062	87 8119	87 8177	87 8234	87 8292	87 8349	87 8407	87 8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	9669	9726	9784	9841	9898	9956	88 0013	88 0070	88 0127	88 0185	57
759	88 0242	88 0299	88 0356	88 0413	88 0471	88 0528	0585	0642	0699	0756	57
760	88 0814	88 0871	88 0928	88 0985	88 1042	88 1099	88 1156	88 1213	88 1271	88 1328	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	88 3661	88 3718	88 3775	88 3832	88 3888	88 3945	88 4002	88 4059	88 4115	88 4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	88 6491	88 6547	88 6604	88 6660	88 6716	88 6773	88 6829	88 6885	88 6942	88 6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	88 9302	88 9358	88 9414	88 9470	88 9526	88 9582	88 9638	88 9694	88 9750	88 9806	56
776	9862	9918	9974	89 0030	89 0086	89 0141	89 0197	89 0253	89 0309	89 0365	56
777	89 0421	89 0477	89 0533	0589	0645	0700	0756	0812	0868	0924	56
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	89 2095	89 2150	89 2206	89 2262	89 2317	89 2373	89 2429	89 2484	89 2540	89 2595	56
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	89 4870	89 4925	89 4980	89 5036	89 5091	89 5146	89 5201	89 5257	89 5312	89 5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	89 7627	89 7682	89 7737	89 7792	89 7847	89 7902	89 7957	89 8012	89 8067	89 8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	90 0039	90 0094	90 0149	90 0203	90 0258	90 0312	55
795	90 0367	90 0422	90 0476	90 0531	90 0586	90 0640	90 0695	90 0749	90 0804	90 0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	90 3090	90 3144	90 3199	90 3253	90 3307	90 3361	90 3416	90 3470	90 3524	90 3578	54
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D.**

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801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	90 5796	90 5850	90 5904	90 5958	90 6012	90 6066	90 6119	90 6173	90 6227	90 6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	90 8485	90 8539	90 8592	90 8646	90 8699	90 8753	90 8807	90 8860	90 8914	90 8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	53
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	91 0037	53
813	91 0091	91 0144	91 0197	91 0251	91 0304	91 0358	91 0411	91 0464	91 0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	91 1158	91 1211	91 1264	91 1317	91 1371	91 1424	91 1477	91 1530	91 1584	91 1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	91 3814	91 3867	91 3920	91 3973	91 4026	91 4079	91 4132	91 4184	91 4237	91 4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	91 6454	91 6507	91 6559	91 6612	91 6664	91 6717	91 6770	91 6822	91 6875	91 6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	91 9078	91 9130	91 9183	91 9235	91 9287	91 9340	91 9392	91 9444	91 9496	91 9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	92 0019	92 0071	52
832	92 0123	92 0176	92 0228	92 0280	92 0332	92 0384	92 0436	92 0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	92 1686	92 1738	92 1790	92 1842	92 1894	92 1946	92 1998	92 2050	92 2102	92 2154	52
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837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	92 4279	92 4331	92 4383	92 4434	92 4486	92 4538	92 4589	92 4641	92 4693	92 4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	92 6857	92 6908	92 6959	92 7011	92 7062	92 7114	92 7165	92 7216	92 7268	92 7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	92 9419	92 9470	92 9521	92 9572	92 9623	92 9674	92 9725	92 9776	92 9827	92 9879	51
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851	9930	9981	93 0032	93 0083	93 0134	93 0185	93 0236	93 0287	93 0338	93 0389	51
852	93 0440	93 0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	93 1966	93 2017	93 2068	93 2118	93 2169	93 2220	93 2271	93 2322	93 2372	93 2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	93 4498	93 4549	93 4599	93 4650	93 4700	93 4751	93 4801	93 4852	93 4902	93 4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	93 7016	93 7066	93 7116	93 7167	93 7217	93 7267	93 7317	93 7367	93 7418	93 7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	93 9519	93 9569	93 9619	93 9669	93 9719	93 9769	93 9819	93 9869	93 9918	93 9968	50
871	94 0018	94 0068	94 0118	94 0168	94 0218	94 0267	94 0317	94 0367	94 0417	94 0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	94 2008	94 2058	94 2107	94 2157	94 2207	94 2256	94 2306	94 2355	94 2405	94 2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
880	94 4483	94 4532	94 4581	94 4631	94 4680	94 4729	94 4779	94 4828	94 4877	94 4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	94 6943	94 6992	94 7041	94 7090	94 7139	94 7189	94 7238	94 7287	94 7336	94 7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	94 9390	94 9439	94 9488	94 9536	94 9585	94 9634	94 9683	94 9731	94 9780	94 9829	49
891	9878	9926	9975	95 0024	95 0073	95 0121	95 0170	95 0219	95 0267	95 0316	49
892	95 0365	95 0414	95 0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	95 1823	95 1872	95 1920	95 1969	95 2017	95 2066	95 2114	95 2163	95 2211	95 2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	95 4243	95 4291	95 4339	95 4387	95 4435	95 4484	95 4532	95 4580	95 4628	95 4677	48
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
900	95 4243	95 4291	95 4339	95 4387	95 4435	95 4484	95 4532	95 4580	95 4628	95 4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	95 6649	95 6697	95 6745	95 6793	95 6840	95 6888	95 6936	95 6984	95 7032	95 7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	95 9041	95 9089	95 9137	95 9185	95 9232	95 9280	95 9328	95 9375	95 9423	95 9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	96 0042	96 0090	96 0138	96 0185	96 0233	96 0280	96 0328	96 0376	96 0423	48
913	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
915	96 1421	96 1469	96 1516	96 1563	96 1611	96 1658	96 1706	96 1753	96 1801	96 1848	48
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	96 3788	96 3835	96 3882	96 3929	96 3977	96 4024	96 4071	96 4118	96 4165	96 4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	96 6142	96 6189	96 6236	96 6283	96 6329	96 6376	96 6423	96 6470	96 6517	96 6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	96 8483	96 8530	96 8576	96 8623	96 8670	96 8716	96 8763	96 8810	96 8856	96 8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	97 0021	97 0068	97 0114	97 0161	97 0207	97 0254	97 0300	47
934	97 0347	97 0393	97 0440	0486	0533	0579	0626	0672	0719	0765	46
935	97 0812	97 0858	97 0904	97 0951	97 0997	97 1044	97 1090	97 1137	97 1183	97 1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
940	97 3128	97 3174	97 3220	97 3266	97 3313	97 3359	97 3405	97 3451	97 3497	97 3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	97 5432	97 5478	97 5524	97 5570	97 5616	97 5662	97 5707	97 5753	97 5799	97 5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	97 7724	97 7769	97 7815	97 7861	97 7906	97 7952	97 7998	97 8043	97 8089	97 8135	46
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D.**

N.	0	1	2	3	4	5	6	7	8	9	D.
950	97 7724	97 7769	97 7815	97 7861	97 7906	97 7952	97 7998	97 8043	97 8089	97 8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	98 0003	98 0049	98 0094	98 0140	98 0185	98 0231	98 0276	98 0322	98 0367	98 0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	98 2271	98 2316	98 2362	98 2407	98 2452	98 2497	98 2543	98 2588	98 2633	98 2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	98 4527	98 4572	98 4617	98 4662	98 4707	98 4752	98 4797	98 4842	98 4887	98 4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	98 6772	98 6817	98 6861	98 6906	98 6951	98 6996	98 7040	98 7085	98 7130	98 7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	98 9005	98 9049	98 9094	98 9138	98 9183	98 9227	98 9272	98 9316	98 9361	98 9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	99 0028	99 0072	99 0117	99 0161	99 0206	99 0250	99 0294	44
978	99 0339	99 0383	99 0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	99 1226	99 1270	99 1315	99 1359	99 1403	99 1448	99 1492	99 1536	99 1580	99 1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	99 3436	99 3480	99 3524	99 3568	99 3613	99 3657	99 3701	99 3745	99 3789	99 3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	99 5635	99 5679	99 5723	99 5767	99 5811	99 5854	99 5898	99 5942	99 5986	99 6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	99 7823	99 7867	99 7910	99 7954	99 7998	99 8041	99 8085	99 8129	99 8172	99 8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D.**

PRACTICAL MATHEMATICS

ANSWERS TO EXAMINATION QUESTIONS

Section 1—Page 25*

1. (a) Five hundred one; (b) Seven thousand eight hundred eight; (c) Thirteen thousand twenty; (d) Seven hundred thirty-four million, five hundred seven thousand. **2.** 27, 123, 3,039 10,006 2,004,112. **3.** (a) XIX XXXI XL (b) $\overline{\text{XX}}$ (c) 1,000. **4.** *Product* is the result of multiplication. Example: $14 \times 4 = 56$ (product). **5.** (a) 414; (b) 3,491; (c) 27,180. **6.** (a) 878; (b) 894; (c) 479; (d) 101; (e) 615. **7.** (a) 376,250; (b) 374,000; (c) 12,377,246. **8.** (a) 189; (b) 320. **9.** 3,249. **10.** 9,040. **11.** 5,453 ft. **12.** 762 acres. **13.** $2,775. **14.** 6 lamps. **15.** 274, 259, 295, 292, 274. **16.** 17 cars. **17.** 40 cows; $60 left. **18.** 842 gal. **19.** $410. **20.** $336.

Section 2—Page 52*

1. 3, 5, 7, 9, 11, 13, 15, 17, 19. **2.** 20, 22, 24, 26, 28, 30. **3.** 19, 23, 29, 31. **4.** 2, 5, 3, 7 and 11. **5.** 2, 2, 3, 11 and 19. **6.** 2. **7.** 2. **8.** 120. **9.** $\frac{2}{9}$. **10.** 40. **11.** $52\frac{1}{2}$. **12.** 7, 7 and 11. **13.** 2, 3, 4, 6 and 12. **14.** 7. **15.** 2.

Section 3—Page 105*

1. $\frac{87}{231}$, $\frac{30}{75}$, $\frac{42}{216}$, $\frac{216}{279}$. **2.** L.C.D. 315. **3.** L.C.D. 24, $\frac{21}{24}$, $\frac{18}{24}$, $\frac{10}{24}$, $\frac{20}{24}$. **4.** L.C.D. 30, $\frac{20}{30}$, $\frac{21}{30}$, $\frac{24}{30}$, $\frac{8}{30}$. **5.** (a) $\frac{2}{3}$; (b) $\frac{4}{5}$; (c) $\frac{13}{15}$. **6.** $12\frac{5}{8}$, $20\frac{5}{6}$, $26\frac{2}{7}$, $24\frac{4}{5}$. **7.** $\frac{27}{4}$, $\frac{65}{12}$, $\frac{80}{9}$, $\frac{171}{16}$. **8.** (a) $18\frac{41}{143}$; (b) $9\frac{1}{3}$. **9.** (a) $1\frac{21}{32}$; (b) $1\frac{21}{60}$ or $1\frac{7}{20}$. **10.** $4\frac{5}{6}$ ft. or 4 ft. 10 in.

Section 4—Page 145*

1. (a) $2\frac{1}{3}$; (b) $1\frac{1}{5}$. **2.** (a) $\frac{1}{4}$; (b) 15. **3.** (a) $3\frac{1}{3}$; (b) $\frac{760}{2109}$ or $\frac{40}{111}$. **4.** $18\frac{75}{116}$. **5.** $487\frac{1}{2}$. **6.** 855. **7.** $26\frac{13}{16}$. **8.** $6,000. **9.** 60 yrs. old. **10.** $250\frac{3}{25}$ mi.

Section 5—Page 175*

1. (a) 288.88888; (b) 94.624. **2.** (a) .21104; (b) 142.2802. **3.** (a) 112; (b) .3737+. **4.** $35\frac{3}{4}$, $745\frac{1}{8}$, $\frac{5}{16}$. **5.** 8.375, 27.5625, 15.09375. **6.** 9.837. **7.** (a) 240 qts.; (b) $20.40 cost. **8.** 28.26. **9.** 4.393+. **10.** $223.80.

Section 6—Page 231*

1. $7,882.72. **2.** $56\frac{1}{4}\%$, $9\frac{3}{8}\%$, $28\frac{1}{8}\%$. **3.** $\frac{7}{16}$, $\frac{5}{32}$, $\frac{7}{8}$. **4.** (a) 1623; (b) 147. **5.** 80 ft. left. **6.** $37.50. **7.** 1,041.7 cu. in. **8.** (a) 10¢ raise per hour; (b) $28\frac{4}{7}\%$. **9.** $264.60 amount paid. **10.** $654.50. **11.** $80 commission. **12.** $66\frac{2}{3}\%$.

*Note—For page numbers, see foot of pages.

Section 7—Page 280*

1. 10 acres. **2.** 253,065.2 sq. mm. difference. **3.** 10 acres. **4.** 201,458 in **5.** 15 weeks, 4 days, 9 hours and 40 minutes. **6.** 12,405 cu. ft. **7.** \$1,750.00 cost of fence. **8.** 1,400 sq. ft. **9.** 41°F. **10.** \$12.50 or 50% profit.

Section 8—Page 302*

1. (a) 186,624; (b) $\frac{64}{343}$. **2.** (a) 6.8121; (b) .000032768. **3.** (a) 49; (b) 34.35. **4.** (a) 1.3863+; (b) .0314. **5.** 264 yds. = 1 side, 1,056 yd. for 4 sides. **6.** 9.4077 feet. **7.** 12 ft. **8.** 45 ft.

Section 9—Page 341*

1. $18\frac{3}{4}$. **2.** $\frac{5}{144}$. **3.** (a) 30; (b) 9. **4.** (a) 7; (b) 1. **5.** (a) $\frac{1}{4}$ or .25; (b) 2. **6.** \$246.75. **7.** $4\frac{4}{35}$. **8.** 225. **9.** $2\frac{1}{2}$ hrs. **10.** 2,790 bricks.

Section 10—Page 388*

1. (a) $A=\pi\times(R^2-r^2)$; (b) $V=a\times h$. **2.** (a) $x=3$; (b) $x=4$; (c) $x=8$; (d) $x=4$. **3.** $P=102\frac{1}{2}$ or 102.5. **4.** No. The *2* cannot be canceled into 8. This is because $11+8y$ is a unit and must be solved first before cancellation can be done; *2* and *8y* are unlike terms, and unlike terms cannot be canceled. **5.** 3,518.592. **6.** .48. **7.** 1.846+ or $1\frac{11}{13}$. **8.** 8750.

Section 11—Page 429*

1. Answer is a graph. **2.** (a) 96, 30, 15; (b) 35, $7\frac{2}{15}$ times. **3.** Answer is a graph. **4.** (a) 450 millions; (b) 110 millions. **5.** Answer is a graph. **6.** Answer is a graph. **7.** (a) 6th yr. \$240; (b) 12th yr. \$156; (c) about the same, steady. **8.** Answer is a graph.

Section 12—Page 480*

1. 242 ft. **2.** 80 sq. yds. **3.** 20 ft. **4.** 22.9 in. or 1.909 ft. or $1\frac{10}{11}$ ft. **5.** 1,600 sq. rds. or 10 acres. **6.** \$17.72. **7.** 4128 sq. in. or $28\frac{2}{3}$ sq. ft. **8.** $14\frac{2}{7}$ yds. or $42\frac{6}{7}$ ft. **9.** 10.392 sq. in. **10.** 38.5 sq. in.

Section 13—Page 524*

1. 161.7 cu. ft. **2.** 216 sq. in. **3.** 59.81 cu. ft. **4.** 2550 sq. ft. **5.** 3.56 ft. **6.** 561.60 cu. ft. **7.** 4,226,981.76 cu. in. **8.** 69.83 cu. in. **9.** (a) $87\frac{1}{2}$¢; (b) $12\frac{1}{2}$¢. **10.** 103.847 lbs.

Section 14—Page 569*

1. 1469.33. **2.** 1101.047. **3.** 1.43612. **4.** .955435. **5.** 493.026. **6.** 8.31881. **7.** .130304. **8.** 54.7409. **9.** 54.4332. **10.** 41.899.

*Note—For page numbers, see foot of pages.